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lim +, δ^+, and Non-Permutability of β-Steps

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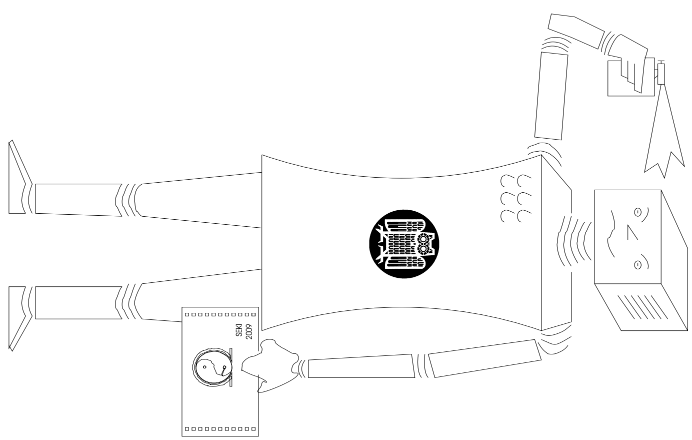

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Abstract

Using a human-oriented formal example proof of the $(\lim +)$ theorem, i.e. that the sum of limits is the limit of the sum, which is of value for reference on its own, we exhibit a non-permutability of β-steps and δ^+-steps (according to Smullyan's classification), which is not visible with non-liberalized δ-rules and not serious with further liberalized δ-rules, such as the δ^{+^+}-rule. Besides a careful presentation of the search for a proof of $(\lim +)$ with several pedagogical intentions, the main subject is to explain why the order of β-steps plays such a practically important role in some calculi.

Contents

1 Motivation

In December 2004, in the theoretical part of an advanced senior-level lecture course [4] on mathematics assistance systems, I presented a formal example proof in a human-oriented sequent calculus that the sum of limits is the limit of the sum $(\lim +)$. *Mathematics assistance systems* are human-oriented interactive theorem provers with strong automation support, aiming at a synergetic interplay between mathematician and machine. PVS [26], ΩMEGA [35], ISABELLE/HOL [23, 27], and QUODLIBET [6] are some of the systems approaching this long term goal.

Considering *reductive calculi* such as sequent, tableau, or matrix calculi, one of the functions of my lectures within the course was to show that—although *sequents* are easier to understand due to their locality—*matrixes* (or *indexed formula trees* [2, 37]) are not only a clever implementation, but—more importantly for us—also needed to follow the proof organization of a working mathematician. To this end, I tried to give the students an idea of the premature commitments forced by sequent and tableau calculi, which require a mathematician to deviate from his intended proof plans and proof-search heuristics.

In his fascinating book [37], Lincoln A. Wallen had criticized the non-permutability of γ- and δ-steps in sequent calculi, according to Raymond M. Smullyan's classification and uniform notation of reductive inference rules as α, β, γ, and δ [36]. I explained how this non-permutability can be overcome by replacing the (non-liberalized) δ-rule (which we will call δ^-*-rule*) with the liberalized δ^+-rule [18]. Along the $(\lim +)$ proof, I then showed that with the δ^+-rule, however, another non-permutability becomes visible, now of the β- and δ^+-steps. Before the liberalization took place to make logicians glad, *this non-permutability was hidden behind the non-permutability of the* γ*- and* δ^-*-steps.*[1]

At that moment, the best logician among my co-lecturers contradicted the occurrence of this non-permutability, and insisted on his opinion when I repeated the material for an introduction in the next lecture. Thus, the non-permutability problems of β-steps deserve publication. A referee of a previous version of this paper called this "an interesting but not too surprising result". Besides this hard result, following the lecture, in this paper we will address some soft aspects of formal calculi for human–machine interaction and publish (for the first time?) a more or less readable, complete, and human-oriented proof of a mathematical standard theorem in a standard general-purpose formal calculus in § 4. We discuss the non-permutabilities of this example proof in § 5, prove the non-permutability of its crucial β- and δ^+-step in § 6, and conclude with an emphasis on open problems in § 7.

Zuerſt werden die Leute eine Sache leugnen; dann werden ſie ſie verharmloſen;
dann werden ſie beſchließen, ſie ſei ſeit langem bekannt.

— ALEXANDER VON HUMBOLDT (cited according to [34], p. x)

2 Introduction to Non-Permutabilities &c.

As explained in [37], the search space of sequent or tableau calculi may suffer from the following weaknesses in design: *Irrelevance*, *Notational Redundancy*, and *Non-Permutability*. Unless explicitly stated otherwise, the weaknesses described in the following apply to sequent and tableau calculi alike.

Irrelevance means, e.g., that when proving the sequent

$$A,\ \ \neg(B \wedge \mathsf{Loves}(\mathsf{Romeo}, y_0^\gamma)),\ \ \mathsf{Loves}(\mathsf{Romeo}, \mathsf{Juliet})$$

with A and B some big formulas, we may try to prove A or $\neg B$ for a long time, although this is not relevant if they are false. Note that in this paper *sequents* are just lists of formulas, i.e. the simplest form that will do for two-valued logics. We call *free γ-variables* (after the γ-steps, which may introduce new ones) (written as y_0^γ) what has the standard names of "meta" [23] or "free" [14] variables. Indeed, free γ-variables must be distinguished from the true meta-variables and the other kinds of free variables we will need. The means to avoid irrelevance is focusing on *connections*, just as the one between $\neg\mathsf{Loves}(\mathsf{Romeo}, y_0^\gamma)$ and $\mathsf{Loves}(\mathsf{Romeo}, \mathsf{Juliet})$. In practice of mathematics assistance systems, however, it is often necessary to expand connectionless parts to support the speculation of lemmas, which then provide a "connection" that is not syntactically obvious, but closes the branch nevertheless. This is especially the case for inductive theorem proving for theoretical [20] and practical [30, 31, 32] reasons.

Notational Redundancy means in a sequent-calculus proof that the offspring sequents repeat the formulas of their ancestor sequents again and again. This is partly overcome in the corresponding tableau calculi. But even tableau proofs repeat the subformulas of their *principal formulas* as *side formulas* [15] again and again. *Structure sharing* can overcome this redundancy and does not differ much for sequent, tableau, or matrix calculi because information on branch, γ-multiplicity, and fairness has to be stored anyway. As mathematics assistance systems are still far from delivering what they once promised to achieve, this optimization is, however, not of top priority, especially because structure sharing is not trivial, but likely to block other improvements: Note that *γ-step multiplicity* requires variable renaming and that different rewrite steps may be applied to the multiple occurrences of subformulas.[2]

Non-Permutability is the subject of this paper. Very roughly speaking, it means that the *order* of inference *steps* (i.e. applications of reductive inference rules) may be crucial for a proof to succeed. Roughly speaking, permutability of two steps S_1 and S_0 simply means the following: *In a closed proof tree where S_0 precedes S_1 and where S_1 was already applicable before S_0, we can do the step S_1 before S_0 and find a closed proof tree nevertheless.* When several formulas in a sequent classify as principal formulas of α-, β-, γ-, or δ-steps, the search space is typically non-confluent. Therefore, a bad order of application of these inference steps may require the search procedure to backtrack or to construct a proof on a higher level of γ-multiplicity than necessary or than a mathematician would expect. Notice that the latter gives a human user hardly any chance to cooperate in proof construction: Who would tell the system to apply a lemma twice when he knows that one application suffices?

When we do a γ-step first and a δ-step second, a proof may fail on the given level of γ-multiplicity, whereas it succeeds when we apply the δ-step first and the γ-step second. For

sequent calculi without free variables (cf. e.g. [15]) this is exemplified in [37, Chapter 1, § 4.3.2]. The reason for this non-permutability is simply that, for the first alternative, due to the eigenvariable condition, the γ-step cannot instantiate its side formula with the parameter introduced by the δ-step.

This non-permutability is not overcome with the introduction of free γ-variables, resulting in the so-called "free-variable" calculi [14, 42]: The reason now is that, for the first alternative, the *variable-condition* blocks the free γ-variable y^{γ} introduced by the γ-step against the instantiation of any term containing the *free* δ^{-}*-variable* $x^{\delta^{-}}$ introduced by the δ^{-}-step. In Skolemizing inference systems, however, we would have to say that y^{γ} becomes an argument of the Skolem term $x^{\delta^{-}}(\ldots y^{\gamma}\ldots)$ introduced by the δ^{-}-step, which causes unification of y^{γ} and $x^{\delta^{-}}(\ldots y^{\gamma}\ldots)$ to fail by the occur check.

This non-permutability is overcome in [37, Chapter 2] with a matrix calculus which generates variable-conditions equivalent to *Outer Skolemization*. As a δ^{+}-step [18] extends the variable-condition only equivalently to *Inner Skolemization* (which is an improvement over Outer Skolemization, i.e. less blockings, or less occurrences in Skolem-terms [24]), this non-permutability is a fortiori overcome by the replacement of the δ^{-}-steps with δ^{+}-steps.

Optimization Problems where a badly chosen order of inference steps does not cause a failure of the proof (at the current level of γ-multiplicity) but only an increase in proof size, are not subsumed under the notion of non-permutability. A typical optimization problem is the following: The size of a proof crucially depends on the β-steps being applied not too early and in the right order. This is obvious from a working mathematician's point of view: Do not start a case analysis before it is needed and make the nested case assumptions in an order that unifies identical argumentations!

Thus, assuming an any-time behavior of a semi-decision procedure for closedness running in parallel (*simultaneous rigid* E*-unification* is not co-semi-decidable [13]), the *folklore heuristics* is somewhat as follows:

Step 1: Apply all α- and δ-steps, guaranteeing termination by deleting their principal formulas from the child sequents (either directly syntactically in sequent calculi, or indirectly by some bookkeeping for search control in tableau calculi).

Step 2: If a γ-rule is applicable to a principal formula that has not reached the current threshold for γ-multiplicity in some branch, do such a γ-step, namely the one with the most promising connections, and then go to Step 1.

Step 3: If a β-rule is applicable, then apply the most promising one, deleting its principal formula from the sequents of the side formulas, and then go to Step 1. Otherwise, if a γ-rule is applicable, then increase the threshold for γ-multiplicity, and then go to Step 2.

3 Background Required for the Example Proof

Before we go on with this abstract expert-style discussion in § 5, we do the proof of $(\lim+)$ in § 4. To this end, we now present a sub-calculus of the calculus of [42], whose development was driven by the integration of Fermat's *descente infinie* into state-of-the-art deduction, with human-orientedness as the second design goal. The calculus uses variable-conditions instead of Skolemization. Variable-conditions are isomorphic to Skolemizaton in the relevant aspects of this paper, but admit the usage of simple variables instead of huge Skolem terms. This improves the readability of our formal proof significantly. We assume the following sets of *variables* to be disjoint:

V_γ	*free γ-variables*, i.e. the free variables of [14]
V_δ	*free δ-variables*, i.e. nullary parameters, instead of Skolem functions
$\mathrm{V}_{\mathrm{bound}}$	*bound variables*, i.e. variables to be bound, cf. below

We use '$\uplus$' for the union of disjoint classes. We partition the free δ-variables into *free δ^--variables* and *free δ^+-variables*: $\mathrm{V}_\delta = \mathrm{V}_{\delta^-} \uplus \mathrm{V}_{\delta^+}$. We define the *free variables* by $\mathrm{V}_{\mathrm{free}} := \mathrm{V}_\gamma \uplus \mathrm{V}_\delta$ and the *variables* by $\mathrm{V} := \mathrm{V}_{\mathrm{bound}} \uplus \mathrm{V}_{\mathrm{free}}$. Finally, the *rigid* variables by $\mathrm{V}_{\gamma\delta^+} := \mathrm{V}_\gamma \uplus \mathrm{V}_{\delta^+}$. We use $\mathcal{V}_k(\Gamma)$ to denote the set of variables from V_k occurring in Γ. We do not permit binding of variables that already occur bound in a term or formula; that is: $\forall x.\ A$ is only a formula if no binder on x already occurs in A. The simple effect is that our formulas are easier to read and our γ- and δ-rules can replace *all* occurrences of x. Moreover, we assume that all binders have minimal scope.

Let σ be a *substitution*. We say that σ is a *substitution on* X if $\mathrm{dom}(\sigma) \subseteq X$. We denote with '$\Gamma\sigma$' the result of replacing each occurrence of a variable $x \in \mathrm{dom}(\sigma)$ in Γ with $\sigma(x)$. Unless otherwise stated, we tacitly assume that all occurrences of variables from $\mathrm{V}_{\mathrm{bound}}$ in a term or formula or in the range of a substitution are *bound occurrences* (i.e. that a variable $x \in \mathrm{V}_{\mathrm{bound}}$ occurs only in the scope of a binder on x) and that each substitution σ satisfies $\mathrm{dom}(\sigma) \subseteq \mathrm{V}_{\mathrm{free}}$, so that no bound occurrences of variables can be replaced and no additional variable occurrences can become bound (i.e. captured) when applying σ.

Definition 3.1 (Variable-Condition, σ-Update, R-Substitution)
A *variable-condition* is a subset of $\mathrm{V}_{\mathrm{free}} \times \mathrm{V}_{\mathrm{free}}$.

Let R be a variable-condition and σ be a substitution. The *σ-update of* R is

$$R \quad \cup \quad \{\, (z^{\mathrm{free}}, x^{\mathrm{free}}) \mid x^{\mathrm{free}} \in \mathrm{dom}(\sigma) \wedge z^{\mathrm{free}} \in \mathcal{V}_{\mathrm{free}}(\sigma(x^{\mathrm{free}})) \,\}.$$

σ is an *R-substitution* if σ is a substitution and the σ-update R' of R is *wellfounded*, i.e. for any nonempty set B, there is a $b \in B$ such that there is no $a \in B$ with $a\ R'\ b$.

Note that, regarding syntax, $(x^{\mathrm{free}}, y^{\mathrm{free}}) \in R$ is intended to mean that an R-substitution σ must not replace x^{free} with a term in which y^{free} could ever occur. This is guaranteed when the σ-updates R' of R are always required to be wellfounded. Indeed, for $z^{\mathrm{free}} \in \mathcal{V}_{\mathrm{free}}(\sigma(x^{\mathrm{free}}))$, we get $z^{\mathrm{free}}\ R'\ x^{\mathrm{free}}\ R'\ y^{\mathrm{free}}$, blocking z^{free} against terms containing y^{free}. In practice, a σ-update of R can always be chosen to be finite. In this case, it is wellfounded iff it is acyclic.

Let A and B be formulas. Let $\mathit{\Gamma}$ and $\mathit{\Pi}$ be sequents, i.e. disjunctive lists of formulas. Let $x \in \mathrm{V}_{\mathrm{bound}}$ be a bound variable, and let $\mathcal{F}$ be the current proof forest, such that $\mathcal{V}(\mathcal{F})$ contains all variables already in use, especially those from $\mathit{\Gamma}$, $\mathit{\Pi}$, and A. Note that $\overline{A}$ is the *conjugate* of the formula A, i.e. B if A is of the form $\neg B$, and $\neg A$ otherwise.

α-rules $\frac{\alpha}{\alpha_0}$:

$$\frac{\mathit{\Gamma}\ \neg\neg A\ \mathit{\Pi}}{A\ \mathit{\Gamma}\ \mathit{\Pi}} \qquad \frac{\mathit{\Gamma}\ (A{\vee}B)\ \mathit{\Pi}}{A\ B\ \mathit{\Gamma}\ \mathit{\Pi}} \qquad \frac{\mathit{\Gamma}\ \neg(A{\wedge}B)\ \mathit{\Pi}}{\overline{A}\ \overline{B}\ \mathit{\Gamma}\ \mathit{\Pi}} \qquad \frac{\mathit{\Gamma}\ (A{\Rightarrow}B)\ \mathit{\Pi}}{\overline{A}\ B\ \mathit{\Gamma}\ \mathit{\Pi}} \qquad \frac{\mathit{\Gamma}\ (A{\Leftarrow}B)\ \mathit{\Pi}}{A\ \overline{B}\ \mathit{\Gamma}\ \mathit{\Pi}}$$

β-rules $\frac{\beta}{\beta_1\ \beta_2}$:

$$\frac{\mathit{\Gamma}\ (A{\wedge}B)\ \mathit{\Pi}}{\begin{array}{c}A\ \mathit{\Gamma}\ \mathit{\Pi}\\ B\ \mathit{\Gamma}\ \mathit{\Pi}\end{array}} \qquad \frac{\mathit{\Gamma}\ \neg(A{\vee}B)\ \mathit{\Pi}}{\begin{array}{c}\overline{A}\ \mathit{\Gamma}\ \mathit{\Pi}\\ \overline{B}\ \mathit{\Gamma}\ \mathit{\Pi}\end{array}} \qquad \frac{\mathit{\Gamma}\ \neg(A{\Rightarrow}B)\ \mathit{\Pi}}{\begin{array}{c}A\ \mathit{\Gamma}\ \mathit{\Pi}\\ \overline{B}\ \mathit{\Gamma}\ \mathit{\Pi}\end{array}} \qquad \frac{\mathit{\Gamma}\ \neg(A{\Leftarrow}B)\ \mathit{\Pi}}{\begin{array}{c}\overline{A}\ \mathit{\Gamma}\ \mathit{\Pi}\\ B\ \mathit{\Gamma}\ \mathit{\Pi}\end{array}}$$

γ-rules $\frac{\gamma}{\gamma_0(t)}$: Let t be any term (by default a new free γ-variable):

$$\frac{\mathit{\Gamma}\quad \exists x.A\quad \mathit{\Pi}}{A\{x{\mapsto}t\}\quad \mathit{\Gamma}\quad \exists x.A\quad \mathit{\Pi}} \qquad \frac{\mathit{\Gamma}\quad \neg\forall x.A\quad \mathit{\Pi}}{\overline{A\{x{\mapsto}t\}}\quad \mathit{\Gamma}\quad \neg\forall x.A\quad \mathit{\Pi}}$$

δ^--rules $\frac{\delta}{\delta^-_0(x^{\delta^-})}$: Let $x^{\delta^-} \in \mathrm{V}_{\delta^-} \setminus \mathcal{V}(\mathcal{F})$ be a new free δ^--variable:

$$\frac{\mathit{\Gamma}\quad \forall x.A\quad \mathit{\Pi}}{A\{x{\mapsto}x^{\delta^-}\}\quad \mathit{\Gamma}\quad \mathit{\Pi}} \qquad \mathcal{V}_{\gamma\delta^+}(\mathit{\Gamma}\ \ \forall x.A\ \ \mathit{\Pi}) \times \{x^{\delta^-}\}$$

$$\frac{\mathit{\Gamma}\quad \neg\exists x.A\quad \mathit{\Pi}}{\overline{A\{x{\mapsto}x^{\delta^-}\}}\quad \mathit{\Gamma}\quad \mathit{\Pi}} \qquad \mathcal{V}_{\gamma\delta^+}(\mathit{\Gamma}\ \ \neg\exists x.A\ \ \mathit{\Pi}) \times \{x^{\delta^-}\}$$

δ^+-rules $\frac{\delta}{\delta^+_0(x^{\delta^+})}$: Let $x^{\delta^+} \in \mathrm{V}_{\delta^+} \setminus \mathcal{V}(\mathcal{F})$ be a new free δ^+-variable:

$$\frac{\mathit{\Gamma}\quad \forall x.A\quad \mathit{\Pi}}{A\{x{\mapsto}x^{\delta^+}\}\quad \mathit{\Gamma}\quad \mathit{\Pi}} \qquad \begin{array}{l}\{(x^{\delta^+},\ \overline{A\{x{\mapsto}x^{\delta^+}\}}\,)\}\\ \mathcal{V}_{\mathrm{free}}(\forall x.A) \times \{x^{\delta^+}\}\end{array}$$

$$\frac{\mathit{\Gamma}\quad \neg\exists x.A\quad \mathit{\Pi}}{\overline{A\{x{\mapsto}x^{\delta^+}\}}\quad \mathit{\Gamma}\quad \mathit{\Pi}} \qquad \begin{array}{l}\{(x^{\delta^+},\ A\{x{\mapsto}x^{\delta^+}\})\}\\ \mathcal{V}_{\mathrm{free}}(\neg\exists x.A) \times \{x^{\delta^+}\}\end{array}$$

Figure 1: The reductive rules of our calculus

3.1 Inference Rules for Reduction Within a Proof Tree

In Figure 1, the inference rules for reductive reasoning within a tree are presented in sequent style. Note that in the good old days when trees grew upwards, Gentzen would have inverted the inference rules such that passing the line means consequence. In our case, passing the line means reduction, and trees grow downwards.

All rules are *sound* and *solution preserving* for the rigid variables in the sense of [42, § 2.4]. Thus, updating a global variable-condition R, we can globally apply any R-substitution on any subset of V_γ without destroying the soundness of the instantiated proof steps.

Instead of an eigenvariable condition, the δ^--rules come with a binary relation on variables to the lower right, which must be added to the current variable-condition R. The δ^+-rules come with an additional relation to the upper right, which has to be added to the *R-choice-condition* C. This choice-condition is an optional part of the calculus. It may store a structure-sharing representation

of an ε-term [19, 16, 41] for a free δ^+-variable, which may restrict the possible values of this variable. As they play only a marginal role in the example proof of § 4, we do not have to discuss choice-conditions here. Note, however, that without a choice-condition, the δ^+-rules would only be sound but not solution preserving, cf. Example 5.1.

Indeed, the calculus contains *different kinds of δ-rules in parallel*. Therefore—to be sound—the δ^--rules have to refer to the the free δ^+-variables introduced by the δ^+-rules in their variable-conditions, and vice versa.

3.2 Lemma Application Between Proof Trees

The reason why we spoke of a proof *forest* $\mathcal{F}$ in Figure 1 is that a proof may be spread over several trees that are connected by generative application of the root of one tree in the reductive proof of another tree, either as a lemma or as an induction hypothesis. While the application of lemmas must be wellfounded, induction hypotheses may be applied to the proof of themselves and mutually. In this paper, we only need lemma application.

Lemma application works as follows. When a lemma $A_1, \ldots, A_m$ is a subsequent of a leaf sequent Γ to be proved (i.e. if, for all $i \in \{1, \ldots, m\}$, the formula A_i is listed in Γ), its application closes the branch of this sequent (*subsumption*). Otherwise, the conjugates of the missing formulas C_i are added to the child sequents (premises), one child per missing formula. This can be seen as Cuts on C_i plus subsumption. More precisely—modulo associativity, commutativity, and idempotency—a sequent $A_1, \ldots, A_m, B_1, \ldots, B_n$ can be reduced by application of the lemma $A_1, \ldots, A_m, C_1, \ldots, C_p$ to the sequents

$$\overline{C_1}, A_1, \ldots, A_m, B_1, \ldots, B_n \qquad \cdots \qquad \overline{C_p}, A_1, \ldots, A_m, B_1, \ldots, B_n.$$

In addition, any time we apply a lemma, we can replace its free δ^--variables locally and arbitrarily, except those free δ^--variables that depend on rigid variables which (in rare cases) may already occur in the input lemma. More precisely, the set of free δ^--variables of a lemma Φ we may instantiate is exactly

$$\left\{\ y^{\delta^-} \in \mathcal{V}_{\delta^-}(\Phi) \ \middle|\ \mathcal{V}_{\gamma\delta^+}(\Phi) \times \{y^{\delta^-}\} \ \subseteq\ R\ \right\}.$$

Typically $\mathcal{V}_{\gamma\delta^+}(\Phi)$ is empty and no restrictions apply. Note that we also may extend this set of free δ^--variables by extending the variable-condition R. This instantiation of outermost δ^--variables mirrors mathematical practice, saves repetition of initial δ-steps, and is essential for induction, where the weights depend on these free δ^--variables to guarantee wellfoundedness. There will be a sufficient number of self-explanatory examples of application of *open lemmas* (i.e. yet unproved lemmas) in § 4.

In the proof below, (2), (3), (4), (5), (6), (7), (8), (9) (where the boxes around the formulas just indicate the matching in the lemma application) and Γ, Ξ, Θ, Ω and σ and t abbreviate the following lemmas and sequents and substitution and term, respectively:

(2): $\min(y^{\delta^-}, z^{\delta^-}) \leq y^{\delta^-}$

(3): $z_4^{\delta^-}{<}z_6^{\delta^-},\ z_4^{\delta^-}{\not<}z_5^{\delta^-},\ z_5^{\delta^-}{\not\leq}z_6^{\delta^-}$

(4): $z_9^{\delta^-}{<}\min(z_{10}^{\delta^-}, z_{11}^{\delta^-}),\ z_9^{\delta^-}{\not<}z_{10}^{\delta^-},\ z_9^{\delta^-}{\not<}z_{11}^{\delta^-}$

(5): $|(z_0^{\delta^-}{+}z_1^{\delta^-}){-}(z_2^{\delta^-}{+}z_3^{\delta^-})| \leq |z_0^{\delta^-}{-}z_2^{\delta^-}| + |z_1^{\delta^-}{-}z_3^{\delta^-}|$

(6): $\boxed{z_4^{\delta^-}{<}z_6^{\delta^-}},\ \boxed{\boxed{z_4^{\delta^-}{\not\leq}z_5^{\delta^-}}},\ z_5^{\delta^-}{\not<}z_6^{\delta^-}$

(7): $z_{12}^{\delta^-}{+}z_{13}^{\delta^-}{<}z_{14}^{\delta^-}{+}z_{15}^{\delta^-},\ \boxed{z_{12}^{\delta^-}{\not<}z_{14}^{\delta^-}},\ \boxed{\boxed{z_{13}^{\delta^-}{\not<}z_{15}^{\delta^-}}}$

(8): $\frac{\varepsilon^{\delta^-}}{2}{+}\frac{\varepsilon^{\delta^-}}{2} \leq \varepsilon^{\delta^-}$

(9): $0{<}\frac{\varepsilon^{\delta^-}}{2},\ 0{\not<}\varepsilon^{\delta^-}$

Γ:

$$\neg\forall\varepsilon_f.\left(0{<}\varepsilon_f \Rightarrow \exists\delta_f{>}0.\ \forall x_f{\neq}x_0^{\delta^-}.\left(\begin{array}{ll} & |f^{\delta^-}(x_f) - y_f^{\delta^-}| < \varepsilon_f \\ \Leftarrow & |x_f{-}x_0^{\delta^-}| < \delta_f \end{array}\right)\right),$$
$$\neg\forall\varepsilon_g.\left(0{<}\varepsilon_g \Rightarrow \exists\delta_g{>}0.\ \forall x_g{\neq}x_0^{\delta^-}.\left(\begin{array}{ll} & |g^{\delta^-}(x_g) - y_g^{\delta^-}| < \varepsilon_g \\ \Leftarrow & |x_g{-}x_0^{\delta^-}| < \delta_g \end{array}\right)\right),$$
$$\exists\delta.\left(0{<}\delta \wedge \forall x{\neq}x_0^{\delta^-}.\left(\begin{array}{ll} & |(f^{\delta^-}(x){+}g^{\delta^-}(x)) - (y_f^{\delta^-}{+}y_g^{\delta^-})| < \varepsilon^{\delta^-} \\ \Leftarrow & |x{-}x_0^{\delta^-}| < \delta \end{array}\right)\right)$$

Ξ:

$$0{<}\delta^{\gamma} \wedge \forall x{\neq}x_0^{\delta^-}.\left(\begin{array}{ll} & |(f^{\delta^-}(x){+}g^{\delta^-}(x)){-}(y_f^{\delta^-}{+}y_g^{\delta^-})| < \varepsilon^{\delta^-} \\ \Leftarrow & |x{-}x_0^{\delta^-}| < \delta^{\gamma} \end{array}\right),$$
$$0{\not<}\varepsilon^{\delta^-},\ \Gamma$$

Θ:

$$\neg\left(0{<}\delta_f^{\delta^+}\ \wedge\ \forall x_f{\neq}x_0^{\delta^-}.\left(\begin{array}{ll} & |f^{\delta^-}(x_f){-}y_f^{\delta^-}| < \varepsilon_f^{\gamma} \\ \Leftarrow & |x_f{-}x_0^{\delta^-}| < \delta_f^{\delta^+} \end{array}\right)\right),$$
$$\neg\exists\delta_g.\left(0{<}\delta_g\ \wedge\ \forall x_g{\neq}x_0^{\delta^-}.\left(\begin{array}{ll} & |g^{\delta^-}(x_g){-}y_g^{\delta^-}| < \varepsilon_g^{\gamma} \\ \Leftarrow & |x_g{-}x_0^{\delta^-}| < \delta_g \end{array}\right)\right),$$
$$0{\not<}\varepsilon^{\delta^-},\ \Gamma$$

Ω:

$$0{\not<}\delta_f^{\delta^+},\ \neg\forall x_f{\neq}x_0^{\delta^-}.\left(\begin{array}{ll} & |f^{\delta^-}(x_f){-}y_f^{\delta^-}| < \varepsilon_f^{\gamma} \\ \Leftarrow & |x_f{-}x_0^{\delta^-}| < \delta_f^{\delta^+} \end{array}\right),$$
$$0{\not<}\delta_g^{\delta^+},\ \neg\forall x_g{\neq}x_0^{\delta^-}.\left(\begin{array}{ll} & |g^{\delta^-}(x_g){-}y_g^{\delta^-}| < \varepsilon_g^{\gamma} \\ \Leftarrow & |x_g{-}x_0^{\delta^-}| < \delta_g^{\delta^+} \end{array}\right),$$
$$0{\not<}\varepsilon^{\delta^-},\ \Gamma$$

σ:

$$\{x_f^{\gamma}{\mapsto}x^{\delta^+},\ x_g^{\gamma}{\mapsto}x^{\delta^+},\ \delta^{\gamma}{\mapsto}\min(\delta_f^{\delta^+}, \delta_g^{\delta^+})\}$$

t:

$$|\ (f^{\delta^-}(x^{\delta^+}){+}g^{\delta^-}(x^{\delta^+})) - (y_f^{\delta^-}{+}y_g^{\delta^-})\ |$$

Figure 2: Global abbreviations for the proof of § 4

4 The $(\lim +)$ Proof: Limit Theorem on Sums in $\mathbb{R}$

4.1 Explanation and Initialization

Compared to the proof of $(\lim +)$ as presented in the lecture courses, the version we present here admits a more rigorous argumentation for non-permutability of β and δ^+ in the following sections.[3]

By standard mathematical abuse of notation, we want to prove the theorem

$$(\lim +) \qquad \lim_{x \to x_0^{\delta^-}} \left(\ f^{\delta^-}(x) + g^{\delta^-}(x) \ \right) = \lim_{x \to x_0^{\delta^-}} f^{\delta^-}(x) + \lim_{x \to x_0^{\delta^-}} g^{\delta^-}(x)$$

Before we start the formal proof, we expand $(\lim +)$ into a better notation:

$$(1): \qquad \left(\begin{array}{rl} & \lim\limits_{x \to x_0^{\delta^-}} f^{\delta^-}(x) = y_f^{\delta^-} \\ \wedge & \lim\limits_{x \to x_0^{\delta^-}} g^{\delta^-}(x) = y_g^{\delta^-} \end{array} \right) \Rightarrow \lim_{x \to x_0^{\delta^-}} \left(\ f^{\delta^-}(x) + g^{\delta^-}(x) \ \right) = y_f^{\delta^-} + y_g^{\delta^-}$$

Warning: The "$=$" here is still no real equality symbol! What is it, then? Something like $\lim\limits_{x \to 0} \left(\ x^2 \sin \frac{1}{x} \ \right) = 0$, formally say $\lim\limits_{x \to z} t_x \ = \ t'$ (*definiendum*), is defined by the formula (*definiens*)

$$\forall \varepsilon{>}0.\ \exists \delta{>}0.\ \forall x{\neq}z.\ \left(\ |t_x - t'| < \varepsilon \ \Leftarrow \ |x - z| < \delta \ \right)$$

Note that $\forall \varepsilon{>}0.\ A$ and $\exists \delta{>}0.\ B$ and $\forall x{\neq}z.\ C$ (*definienda*) abbreviate $\forall \varepsilon.\ (0{<}\varepsilon \Rightarrow A)$ and $\exists \delta.\ (0{<}\delta \wedge B)$ and $\forall x.\ (x{\neq}z \Rightarrow C)$ (*definientia*), respectively. Thus, if—in what follows—we speak of an *expansion of* "$\forall \varepsilon{>}0.\ \ldots$" (from *definiendum* to *definiens*) or simply of *an expansion of* $\forall$, we mean the replacement of $\forall \varepsilon{>}0.\ A$ with $\forall \varepsilon.\ (0{<}\varepsilon \Rightarrow A)$ for some formula A in a reductive proof step. Analogous proof steps are meant by *expansion of* $\exists$ and *expansion of* $\lim$, respectively. We will often reorder the formulas in the sequents without mentioning it.

We initialize our global variable-condition R by $R := \emptyset$, and our global R-choice-condition C by $C := \emptyset$.

4.2 Expanding the Proof Tree with Root (1)

By two α-steps and expansion of $\lim$ from *definiendum* to *definiens*, we reduce (1) to its single child (1.1), writing (1^2) for (1.1):

$$(1^2): \qquad \forall \varepsilon{>}0.\ \exists \delta{>}0.\ \forall x{\neq}x_0^{\delta^-}.\ \left(\begin{array}{rl} & |(f^{\delta^-}(x) + g^{\delta^-}(x)) - (y_f^{\delta^-} + y_g^{\delta^-})| < \varepsilon \\ \Leftarrow & |x - x_0^{\delta^-}| < \delta \end{array} \right),$$
$$\lim_{x \to x_0^{\delta^-}} f^{\delta^-}(x) \neq y_f^{\delta^-}, \quad \lim_{x \to x_0^{\delta^-}} g^{\delta^-}(x) \neq y_g^{\delta^-}$$

By expansion of "$\forall \varepsilon{>}0.\ \ldots$" from *definiendum* to *definiens*, then a δ^-- and an α-step, and finally expansion of $\exists$ and some reordering of the listed formulas we reduce this to:

$$(\mathbf{1^3}): \ \exists \delta.\ \left(\ 0{<}\delta \wedge \forall x{\neq}x_0^{\delta^-}.\ \left(\begin{array}{rl} & |(f^{\delta^-}(x) + g^{\delta^-}(x)) - (y_f^{\delta^-} + y_g^{\delta^-})| < \varepsilon^{\delta^-} \\ \Leftarrow & |x - x_0^{\delta^-}| < \delta \end{array} \right) \right),$$
$$0 \not< \varepsilon^{\delta^-}, \quad \lim_{x \to x_0^{\delta^-}} f^{\delta^-}(x) \neq y_f^{\delta^-}, \quad \lim_{x \to x_0^{\delta^-}} g^{\delta^-}(x) \neq y_g^{\delta^-}$$

A γ-step yields:

$(\mathbf{1^4})$: $$0{<}\delta^\gamma \wedge \forall x{\neq}x_0^{\delta^-}.\left(\begin{array}{rl} & |(f^{\delta^-}(x){+}g^{\delta^-}(x)){-}(y_f^{\delta^-}{+}y_g^{\delta^-})| < \varepsilon^{\delta^-} \\ \Leftarrow & |x{-}x_0^{\delta^-}| < \delta^\gamma \end{array}\right),\ (1^3)$$

Note that the (1^3) at the end of the sequent (1^4) means that the whole parent sequent is part of the child sequent.

Expanding $\lim$ and $\forall$, plus a γ-step, each twice, we get (cf. Figure 2 for $\mathit{\Xi}$):

$(\mathbf{1^5})$: $$\begin{array}{l} \neg\left(0{<}\varepsilon_f^\gamma \Rightarrow \exists\delta_f{>}0.\ \forall x_f{\neq}x_0^{\delta^-}.\left(\begin{array}{rl} & |f^{\delta^-}(x_f){-}y_f^{\delta^-}| < \varepsilon_f^\gamma \\ \Leftarrow & |x_f{-}x_0^{\delta^-}| < \delta_f \end{array}\right)\right), \\ \neg\left(0{<}\varepsilon_g^\gamma \Rightarrow \exists\delta_g{>}0.\ \forall x_g{\neq}x_0^{\delta^-}.\left(\begin{array}{rl} & |g^{\delta^-}(x_g){-}y_g^{\delta^-}| < \varepsilon_g^\gamma \\ \Leftarrow & |x_g{-}x_0^{\delta^-}| < \delta_g \end{array}\right)\right),\ \mathit{\Xi} \end{array}$$

A β-step and an expansion of $\exists$, each twice, yield:

$(\mathbf{1^5.1})$: $$0{<}\varepsilon_f^\gamma,\ \neg\left(0{<}\varepsilon_g^\gamma \Rightarrow \exists\delta_g{>}0.\ \forall x_g{\neq}x_0^{\delta^-}.\left(\begin{array}{rl} & |g^{\delta^-}(x_g){-}y_g^{\delta^-}| < \varepsilon_g^\gamma \\ \Leftarrow & |x_g{-}x_0^{\delta^-}| < \delta_g \end{array}\right)\right),\ \mathit{\Xi}$$

$(\mathbf{1^5.2})$: $$0{<}\varepsilon_g^\gamma,\ \neg\exists\delta_f{>}0.\ \forall x_f{\neq}x_0^{\delta^-}.\left(\begin{array}{rl} & |f^{\delta^-}(x_f){-}y_f^{\delta^-}| < \varepsilon_f^\gamma \\ \Leftarrow & |x_f{-}x_0^{\delta^-}| < \delta_f \end{array}\right),\ \mathit{\Xi}$$

$(\mathbf{1^5.3})$: $$\begin{array}{l} \neg\exists\delta_f.\left(0{<}\delta_f\ \wedge\ \forall x_f{\neq}x_0^{\delta^-}.\left(\begin{array}{rl} & |f^{\delta^-}(x_f){-}y_f^{\delta^-}| < \varepsilon_f^\gamma \\ \Leftarrow & |x_f{-}x_0^{\delta^-}| < \delta_f \end{array}\right)\right), \\ \neg\exists\delta_g.\left(0{<}\delta_g\ \wedge\ \forall x_g{\neq}x_0^{\delta^-}.\left(\begin{array}{rl} & |g^{\delta^-}(x_g){-}y_g^{\delta^-}| < \varepsilon_g^\gamma \\ \Leftarrow & |x_g{-}x_0^{\delta^-}| < \delta_g \end{array}\right)\right),\ \mathit{\Xi} \end{array}$$

A δ^+-step applied to the first formula at $(1^5.3)$ yields:

$(\mathbf{1^5.3.1})$: $$0{<}\delta^\gamma \wedge \forall x{\neq}x_0^{\delta^-}.\left(\begin{array}{rl} & |(f^{\delta^-}(x){+}g^{\delta^-}(x)){-}(y_f^{\delta^-}{+}y_g^{\delta^-})| < \varepsilon^{\delta^-} \\ \Leftarrow & |x{-}x_0^{\delta^-}| < \delta^\gamma \end{array}\right),\ \mathit{\Theta}$$

where R is extended with $\{x_0^{\delta^-}, f^{\delta^-}, y_f^{\delta^-}, \varepsilon_f^\gamma\} \times \{\delta_f^{\delta^+}\}$, and the choice-condition C with:

$$\left\{\ \delta_f^{\delta^+} \mapsto \left(0{<}\delta_f^{\delta^+}\ \wedge\ \forall x_f{\neq}x_0^{\delta^-}.\left(\begin{array}{rl} & |f^{\delta^-}(x_f){-}y_f^{\delta^-}| < \varepsilon_f^\gamma \\ \Leftarrow & |x_f{-}x_0^{\delta^-}| < \delta_f^{\delta^+} \end{array}\right)\right)\ \right\}$$

4.3 A Bad Turn

Now we do an early β-step against the folklore heuristics presented in § 2. This will make the whole following subproof fail! A reader who is interested only in a successful example proof may continue reading with § 4.6.

$(\mathbf{1^5.3.1.1})$: $$0{<}\delta^\gamma,\ \mathit{\Theta}$$

$(\mathbf{1^5.3.1.2})$: $$\forall x{\neq}x_0^{\delta^-}.\left(\begin{array}{rl} & |(f^{\delta^-}(x){+}g^{\delta^-}(x)) - (y_f^{\delta^-}{+}y_g^{\delta^-})| < \varepsilon^{\delta^-} \\ \Leftarrow & |x{-}x_0^{\delta^-}| < \delta^\gamma \end{array}\right),\ \mathit{\Theta}$$

A δ^+-step, two α-steps, and expansion of $\forall$, applied to $(1^5.3.1.2)$, yield:

$(\mathbf{1^5.3.1.2.1})$: $$\forall x.\ \left(\ x{\neq}x_0^{\delta^-} \Rightarrow \left(\begin{array}{ll} & |(f^{\delta^-}(x){+}g^{\delta^-}(x)) - (y_f^{\delta^-}{+}y_g^{\delta^-})| < \varepsilon^{\delta^-} \\ \Leftarrow & |x{-}x_0^{\delta^-}| < \delta^{\gamma}\end{array}\right)\right),\ \mathit{\Omega}$$

where R is extended with $\{x_0^{\delta^-}, g^{\delta^-}, y_g^{\delta^-}, \varepsilon_g^{\gamma}\} \times \{\delta_g^{\delta^+}\}$, and C with:

$$\left\{\ \delta_g^{\delta^+} \mapsto \left(\ 0{<}\delta_g^{\delta^+}\ \wedge\ \forall x_g{\neq}x_0^{\delta^-}.\ \left(\begin{array}{ll} & |g^{\delta^-}(x_g){-}y_g^{\delta^-}| < \varepsilon_g^{\gamma} \\ \Leftarrow & |x_g{-}x_0^{\delta^-}| < \delta_g^{\delta^+}\end{array}\right)\right)\ \right\}$$

A δ^+-step and two α-steps yield (cf. Figure 2 for t):

$(1^5.3.1.2.1^2)$: $$x^{\delta^+}{=}x_0^{\delta^-},\ t < \varepsilon^{\delta^-},\ |x^{\delta^+}{-}x_0^{\delta^-}| \not< \delta^{\gamma},\ \mathit{\Omega}$$

where R is extended with $\{x_0^{\delta^-}, f^{\delta^-}, g^{\delta^-}, y_f^{\delta^-}, y_g^{\delta^-}, \varepsilon^{\delta^-}, \delta^{\gamma}\} \times \{x^{\delta^+}\}$
and our R-choice-condition C with

$$\left\{\ x^{\delta^+} \mapsto \neg\left(\ x^{\delta^+}{\neq}x_0^{\delta^-} \Rightarrow \left(\ t < \varepsilon^{\delta^-}\ \Leftarrow\ |x^{\delta^+}{-}x_0^{\delta^-}| < \delta^{\gamma}\ \right)\right)\ \right\}$$

Expansion of $\forall$ and a γ-step, each twice, yield:

$(1^5.3.1.2.1^3)$: $$\begin{array}{l} \neg\left(\ x_f^{\gamma}{\neq}x_0^{\delta^-}\ \Rightarrow \left(\begin{array}{ll} & |f^{\delta^-}(x_f^{\gamma}){-}y_f^{\delta^-}| < \varepsilon_f^{\gamma} \\ \Leftarrow & |x_f^{\gamma}{-}x_0^{\delta^-}| < \delta_f^{\delta^+}\end{array}\right)\right), \\ \neg\left(\ x_g^{\gamma}{\neq}x_0^{\delta^-}\ \Rightarrow \left(\begin{array}{ll} & |g^{\delta^-}(x_g^{\gamma}){-}y_g^{\delta^-}| < \varepsilon_g^{\gamma} \\ \Leftarrow & |x_g^{\gamma}{-}x_0^{\delta^-}| < \delta_g^{\delta^+}\end{array}\right)\right), \\ x^{\delta^+}{=}x_0^{\delta^-},\ t < \varepsilon^{\delta^-},\ |x^{\delta^+}{-}x_0^{\delta^-}| \not< \delta^{\gamma},\ \mathit{\Omega} \end{array}$$

4.4 Partial Success

2 β-steps, each twice, yield:

$(1^5.3.1.2.1^3.1)$: $x_f^{\gamma}{\neq}x_0^{\delta^-},\ x^{\delta^+}{=}x_0^{\delta^-},\ \ldots$

$(1^5.3.1.2.1^3.2)$: $x_g^{\gamma}{\neq}x_0^{\delta^-},\ x^{\delta^+}{=}x_0^{\delta^-},\ \ldots$

$(1^5.3.1.2.1^3.3)$: $|x_f^{\gamma}{-}x_0^{\delta^-}| < \delta_f^{\delta^+},\ |x^{\delta^+}{-}x_0^{\delta^-}| \not< \delta^{\gamma},\ \ldots$

$(1^5.3.1.2.1^3.4)$: $|x_g^{\gamma}{-}x_0^{\delta^-}| < \delta_g^{\delta^+},\ |x^{\delta^+}{-}x_0^{\delta^-}| \not< \delta^{\gamma},\ \ldots$

$(1^5.3.1.2.1^3.5)$: $|f^{\delta^-}(x_f^{\gamma}) - y_f^{\delta^-}| \not< \varepsilon_f^{\gamma},\ |g^{\delta^-}(x_g^{\gamma}) - y_g^{\delta^-}| \not< \varepsilon_g^{\gamma},$
$$x^{\delta^+}{=}x_0^{\delta^-},\ t < \varepsilon^{\delta^-},\ |x^{\delta^+}{-}x_0^{\delta^-}| \not< \delta^{\gamma},\ \mathit{\Omega}$$

And now? By formula unification and some basic knowledge of the domain, we can easily see that global application of the substitution σ from § 4.1 admits to close the branches of the first four sequents. According to Definition 3.1, this adds
$$\{(x^{\delta^+}, x_f^{\gamma}), (x^{\delta^+}, x_g^{\gamma}), (\delta_f^{\delta^+}, \delta^{\gamma}), (\delta_g^{\delta^+}, \delta^{\gamma})\}$$
to our variable-condition R, which, luckily, stays acyclic, cf. the acyclic graph of Figure 5 in § 4.8. $(1^5.3.1.2.1^3.1)$ and $(1^5.3.1.2.1^3.2)$ become logical axioms. Applying lemma (2) of Figure 2 instantiated via $\{y^{\delta^-}{\mapsto}\delta_f^{\delta^+},\ z^{\delta^-}{\mapsto}\delta_g^{\delta^+}\}$ we reduce $(1^5.3.1.2.1^3.3)$ to:

$(1^5.3.1.2.1^3.3.1)$: $\min(\delta_f^{\delta^+}, \delta_g^{\delta^+}) \not\leq \delta_f^{\delta^+},\ |x^{\delta^+}{-}x_0^{\delta^-}| < \delta_f^{\delta^+},\ |x^{\delta^+}{-}x_0^{\delta^-}| \not< \min(\delta_f^{\delta^+}, \delta_g^{\delta^+}),\ \ldots$

which is subsumed by the transitivity lemma (3) of Figure 2.
$(1^5.3.1.2.1^3.4)$ can be closed analogously to $(1^5.3.1.2.1^3.3)$.

$$
\begin{tikzcd}[column sep=small]
 & (1) \arrow[d, "1^2"', "{\alpha_0^2,\ \lim,\ \forall,\ \delta_0^-(\varepsilon^{\delta^-}),\ \alpha_0,\ \exists}"] & \\
 & (1^3) \arrow[d, "1"', "{\gamma_0(\min(\delta_f^{\delta^+},\delta_g^{\delta^+}))}"] & \\
 & (1^4) \arrow[d, "1"', "{\lim^2,\ \forall^2,\ \gamma_0(\varepsilon_f^\gamma),\ \gamma_0(\varepsilon_g^\gamma)}"] & \\
 & (1^5) \arrow[dl, "1"', "\beta_1"] \arrow[d, "2"', "{\beta_2,\ \beta_1}"] \arrow[dr, "{\beta_2^2,\ \exists^2}", "3"'] & \\
(1^5.1) & (1^5.2) & (1^5.3) \arrow[d, "1"', "{\delta_0^+(\delta_f^{\delta^+})}"] \\
 & & (1^5.3.1) \arrow[dl, "1"', "\beta_1"] \arrow[d, "2"', "\beta_2"] \\
 & (1^5.3.1.1) & (1^5.3.1.2) \arrow[d, "1"', "{\delta_0^+(\delta_g^{\delta^+}),\ \alpha_0^2,\ \forall}"] \\
 & & (1^5.3.1.2.1) \arrow[d, "1"', "{\delta_0^+(x^{\delta^+}),\ \alpha_0^2}"] \\
 & & (1^5.3.1.2.1^2) \arrow[d, "1"', "{\forall^2,\ \gamma_0(x^{\delta^+})^2}"] \\
 & & (1^5.3.1.2.1^3)
\end{tikzcd}
$$

Figure 3: Non-Permutability of β at $(1^5.3.1)$ and δ^+ at $(1^5.3.1.2)$:
No chance to prove $0 < \min(\delta_f^{\delta^+}, \delta_g^{\delta^+})$ at $(1^5.3.1.1)$

4.5 Total Failure

Abstractly, our proof tree looks as in Figure 3. By the application of σ, $(1^5.3.1.1)$ has become

$$0 < \min(\delta_f^{\delta^+}, \delta_g^{\delta^+}),\ \Theta$$

If the first formula—which is the only new one as compared to its parent sequent—is irrelevant for the proof of $(1^5.3.1.1)$ (in the sense that it is not contributing as a principal formula, cf. [15, 30, 32]), then we had better prove $(1^5.3.1)$ instead, because this saves us the proof of the whole β_2-subtree of $(1^5.3.1)$. But look: $\delta_g^{\delta^+}$ is not introduced before $(1^5.3.1.2.1)$, which in $(1^5.3.1.2.1^2)$ results in the context $0 \not< \delta_f^{\delta^+},\ 0 \not< \delta_g^{\delta^+}$ (as listed in Ω of Figure 2) with which we could prove $0 < \min(\delta_f^{\delta^+}, \delta_g^{\delta^+})$ by lemma (4) of Figure 2. Thus, the β-step applied to $(1^5.3.1)$ does not have any benefit unless it is done *below* $(1^5.3.1.2.1)$.

Now, we have three possibilities in principle:

1. We can backtrack to $(1^5.3.1)$, deleting all its sub-trees.

2. We could try to use the choice-condition of $\delta_g^{\delta^+}$ to find out that it is positive. $C(\delta_g^{\delta^+})$ is
$$0{<}\delta_g^{\delta^+} \;\wedge\; \forall x_g{\neq}x_0^{\delta^-}.\ \big(\ |g^{\delta^-}(x_g){-}y_g^{\delta^-}| < \varepsilon_g^{\gamma} \;\Leftarrow\; |x_g{-}x_0^{\delta^-}| < \delta_g^{\delta^+}\ \big).$$
But this guarantees $0{<}\delta_g^{\delta^+}$ only if also the second part of the conjunction can be shown to be satisfiable, for which we again lack the context.

3. We can prove $(1^5.3.1.1)$ by proving its subsequent Θ. As Θ is already a subsequent of $(1^5.3.1)$, this means that we could prove already $(1^5.3.1)$ this way. Thus, the whole subproof below $(1^5.3.1.2)$ could be pruned. Moreover, as we would have to expand the principal γ-formula of (1^3) a second time, resulting in a higher maximum of γ-multiplicity than necessary, the following lemma holds.

Lemma 4.1 *Using the reductive rules of Figure 1 with a γ-multiplicity threshold of 1, the current proof tree (with the partial instantiation σ) cannot be expanded and instantiated to a closed proof tree at $(1^5.1)$, $(1^5.2)$, and $(1^5.3.1.1)$ in parallel.*

For a proof of Lemma 4.1 cf. § 6.1. Note that the validity of Lemma 4.1 depends on the δ^-- and δ^+-rules being the only δ-rules available. With δ^{+^+}-rules the situation would be different, cf. § 5.4. Moreover, as our proof trees are customary AND-trees (and no AND/OR-trees that admit alternative proof attempts as in [5, 6]), Lemma 4.1 means that the whole proof attempt is failed for a γ-multiplicity of 1.

4.6 Backtracking to the Path of Virtue

Item 1 in the above list is the only reasonable alternative. Therefore, let us restart from $(1^5.3.1)$ —not without storing σ and its connections before.

Applied to $(1^5.3.1)$, one δ^+-step, two α-steps, two expansions of $\forall$, and two γ-steps yield as in § 4.3 and with the same extensions of R and C:

$(1^5.3.1^2)$:
$$\begin{array}{l}
\neg\left(\ x_f^{\gamma}{\neq}x_0^{\delta^-} \;\Rightarrow\; \left(\begin{array}{ll} & |f^{\delta^-}(x_f^{\gamma}){-}y_f^{\delta^-}| < \varepsilon_f^{\gamma} \\ \Leftarrow & |x_f^{\gamma}{-}x_0^{\delta^-}| < \delta_f^{\delta^+}\end{array}\right)\ \right),\\
\neg\left(\ x_g^{\gamma}{\neq}x_0^{\delta^-} \;\Rightarrow\; \left(\begin{array}{ll} & |g^{\delta^-}(x_g^{\gamma}){-}y_g^{\delta^-}| < \varepsilon_g^{\gamma} \\ \Leftarrow & |x_g^{\gamma}{-}x_0^{\delta^-}| < \delta_g^{\delta^+}\end{array}\right)\ \right),\\
0{<}\delta^{\gamma} \wedge \forall x{\neq}x_0^{\delta^-}.\left(\begin{array}{ll} & |(f^{\delta^-}(x){+}g^{\delta^-}(x)){-}(y_f^{\delta^-}{+}y_g^{\delta^-})| < \varepsilon^{\delta^-} \\ \Leftarrow & |x{-}x_0^{\delta^-}| < \delta^{\gamma}\end{array}\right),\ \Omega
\end{array}$$

$$
\begin{tikzcd}[column sep=small]
 & (1^5.3.1^2) \arrow[dl, "\beta_1"'] \arrow[d, "\beta_2"] \\
\boxed{\boxed{x_f^{\gamma} \neq x_0^{\delta^-}}},\ B,\ C,\ \mathit{\Omega} & A,\ B,\ C,\ \mathit{\Omega} \arrow[dl, "\beta_1"'] \arrow[d, "\beta_2"] \\
A,\ B,\ 0 < \delta^{\gamma},\ \mathit{\Omega} & A,\ B,\ \forall x \neq x_0^{\delta^-}.\left(\begin{array}{ll} & |\ldots - (y_f^{\delta^-} + y_g^{\delta^-})| < \varepsilon^{\delta^-} \\ \Leftarrow & |x - x_0^{\delta^-}| < \delta^{\gamma} \end{array} \right),\ \mathit{\Omega} \arrow[d, "{\forall,\ \delta_0^+(x^{\delta^+}),\ \alpha_0^2}"] \\
 & \begin{array}{c} A,\ B,\ \boxed{\boxed{x^{\delta^+} = x_0^{\delta^-}}},\ \boxed{|x^{\delta^+} - x_0^{\delta^-}| \not< \delta^{\gamma}}, \\ t < \varepsilon^{\delta^-},\ \mathit{\Omega} \end{array}
\end{tikzcd}
$$

Here A denotes the formula $\neg\Big(\ |f^{\delta^-}(x_f^{\gamma}) - y_f^{\delta^-}| < \varepsilon_f^{\gamma}\ \Leftarrow\ \boxed{|x_f^{\gamma} - x_0^{\delta^-}| < \delta_f^{\delta^+}}\ \Big)$. B and C denote the second and third β-formula of the sequent $(1^5.3.1^2)$, respectively. And $\mathit{\Pi}$ the sequent at the second (β_2-) child of the root without the second β-formula, i.e. without the third β-formula of $(1^5.3.1^2)$.

Figure 4: Non-Permutability of β at $(1^5.3.1^2)$ and
β at the β_2-child of $(1^5.3.1^2)$:
No chance to prove $x_f^{\gamma} \neq x_0^{\delta^-}$ at leftmost leaf

Now we *have to* expand one of the three first β-formulas of $(1^5.3.1^2)$. Note that the third one is the one whose expansion made our proof fail before. We have learned that the path of virtue is narrow! What about taking the first β-formula? This would result in the subtree depicted in Figure 4 above! Its first β-step can represent progress only if the first (β_1-) child is easier to prove than the root itself. But the only reasonable connection of its single new formula $\boxed{\boxed{x_f^{\gamma} \neq x_0^{\delta^-}}}$ is to the third formula $\boxed{\boxed{x^{\delta^+} = x_0^{\delta^-}}}$ of the rightmost leaf; via σ. Thus, we would have to copy the proof starting below the second (β_2-) child of the root to its first (β_1-) child. But, if we do so, this proof will fail again, due to the following reason: To close the copied subproof we need the connection between the fourth formula $\boxed{|x^{\delta^+} - x_0^{\delta^-}| \not< \delta^{\gamma}}$ of the rightmost leaf and the positive subformula $\boxed{|x_f^{\gamma} - x_0^{\delta^-}| < \delta_f^{\delta^+}}$ of the formula A; via σ, (2), and (3) as at the end of § 4.4. But this connection is only available at the original position and not at the position the subproof is copied to, because the positive subformula is part of the β_2-side formula A of the β-step at the root. All in all, this shows that expanding the first β-formula of $(1^5.3.1^2)$ leads to a failure of the proof on the current threshold for γ-multiplicity again. By symmetry, the same holds for the second. Thus, we take the third. Notice that the β-step we *have to* do now is the one whose too early application made us backtrack before.

A β-step to the third β-formula of $(1^5.3.1^2)$, and expansion of $\forall$ yield:

$(1^5.3.1^2.1)$: $0{<}\delta^{\gamma},\ 0{\not<}\delta^{\delta^+}_{f},\ 0{\not<}\delta^{\delta^+}_{g},\ \ldots$

$(1^5.3.1^2.2)$:
$$\begin{array}{l}\neg\left(\ x^{\gamma}_{f}{\neq}x^{\delta^-}_{0}\ \Rightarrow\ \left(\begin{array}{ll} & |f^{\delta^-}(x^{\gamma}_{f}){-}y^{\delta^-}_{f}| < \varepsilon^{\gamma}_{f}\\ \Leftarrow & |x^{\gamma}_{f}{-}x^{\delta^-}_{0}| < \delta^{\delta^+}_{f}\end{array}\right)\right),\\ \neg\left(\ x^{\gamma}_{g}{\neq}x^{\delta^-}_{0}\ \Rightarrow\ \left(\begin{array}{ll} & |g^{\delta^-}(x^{\gamma}_{g}){-}y^{\delta^-}_{g}| < \varepsilon^{\gamma}_{g}\\ \Leftarrow & |x^{\gamma}_{g}{-}x^{\delta^-}_{0}| < \delta^{\delta^+}_{g}\end{array}\right)\right),\\ \forall x.\left(\ x{\neq}x^{\delta^-}_{0}\Rightarrow\left(\begin{array}{ll} & |(f^{\delta^-}(x){+}g^{\delta^-}(x)){-}(y^{\delta^-}_{f}{+}y^{\delta^-}_{g})| < \varepsilon^{\delta^-}\\ \Leftarrow & |x{-}x^{\delta^-}_{0}| < \delta^{\gamma}\end{array}\right)\right),\ \mathit{\Omega}\end{array}$$

As a δ^--step with the first formula of the last line of $(1^5.3.1^2.2)$ as principal formula would block the later instantiation of x^{γ}_{f} and x^{γ}_{g} with the newly introduced free δ-variable, for the proof to succeed on the current threshold for γ-multiplicity, we have to take a δ^+-step instead. Note that this was not yet a problem for the sequent $(1^5.3.1.2.1)$ of § 4.3, in which x^{γ}_{f} and x^{γ}_{g} did not occur yet. Besides the δ^+-step extending R and C as in § 4.3, we do two α-steps. This results exactly in what was seen before at the end of § 4.3, with the exception of a different label:

$(1^5.3.1^2.2.1)$:
$$\begin{array}{l}\neg\left(\ x^{\gamma}_{f}{\neq}x^{\delta^-}_{0}\ \Rightarrow\ \left(\begin{array}{ll} & |f^{\delta^-}(x^{\gamma}_{f}){-}y^{\delta^-}_{f}| < \varepsilon^{\gamma}_{f}\\ \Leftarrow & |x^{\gamma}_{f}{-}x^{\delta^-}_{0}| < \delta^{\delta^+}_{f}\end{array}\right)\right),\\ \neg\left(\ x^{\gamma}_{g}{\neq}x^{\delta^-}_{0}\ \Rightarrow\ \left(\begin{array}{ll} & |g^{\delta^-}(x^{\gamma}_{g}){-}y^{\delta^-}_{g}| < \varepsilon^{\gamma}_{g}\\ \Leftarrow & |x^{\gamma}_{g}{-}x^{\delta^-}_{0}| < \delta^{\delta^+}_{g}\end{array}\right)\right),\\ x^{\delta^+}{=}x^{\delta^-}_{0},\ t < \varepsilon^{\delta^-},\ |x^{\delta^+}{-}x^{\delta^-}_{0}| \not< \delta^{\gamma},\ \mathit{\Omega}\end{array}$$

Again, two β-steps, each twice, yield:

$(1^5.3.1^2.2.1.1)$: $x^{\gamma}_{f}{\neq}x^{\delta^-}_{0},\ x^{\delta^+}{=}x^{\delta^-}_{0},\ \ldots$

$(1^5.3.1^2.2.1.2)$: $x^{\gamma}_{g}{\neq}x^{\delta^-}_{0},\ x^{\delta^+}{=}x^{\delta^-}_{0},\ \ldots$

$(1^5.3.1^2.2.1.3)$: $|x^{\gamma}_{f}{-}x^{\delta^-}_{0}| < \delta^{\delta^+}_{f},\ |x^{\delta^+}{-}x^{\delta^-}_{0}| \not< \delta^{\gamma},\ \ldots$

$(1^5.3.1^2.2.1.4)$: $|x^{\gamma}_{g}{-}x^{\delta^-}_{0}| < \delta^{\delta^+}_{g},\ |x^{\delta^+}{-}x^{\delta^-}_{0}| \not< \delta^{\gamma},\ \ldots$

$(1^5.3.1^2.2.1.5)$: $|f^{\delta^-}(x^{\gamma}_{f}) - y^{\delta^-}_{f}| \not< \varepsilon^{\gamma}_{f},\ |g^{\delta^-}(x^{\gamma}_{g}) - y^{\delta^-}_{g}| \not< \varepsilon^{\gamma}_{g},$
$$x^{\delta^+}{=}x^{\delta^-}_{0},\ t < \varepsilon^{\delta^-},\ |x^{\delta^+}{-}x^{\delta^-}_{0}| \not< \delta^{\gamma},\ \mathit{\Omega}$$

As before in § 4.4, application of σ admits the closure of of the four branches of $(1^5.3.1^2.2.1.[1–4])$. But now, contrary to what made us backtrack before, $(1^5.3.1^2.1)$ becomes

$$0{<}\min(\delta^{\delta^+}_{f},\delta^{\delta^+}_{g}),\ 0{\not<}\delta^{\delta^+}_{f},\ 0{\not<}\delta^{\delta^+}_{g},\ \ldots,$$

which is subsumed by an instance of lemma (4) of Figure 2.

4.7 A Working Mathematician's Immediate Focus

Note that $(1^5.3.1^2.2.1.5)$ would have been the immediate focus of a working mathematician. He would have sequenced all the lousy β-steps *after* doing the crucial steps of the proof which we can do only now. *Notice that the matrix (indexed formula tree) versions of our calculus will enable us to support this human behavior in the follow-up lectures.* Let us repeat $(1^5.3.1^2.2.1.5)$ with some omissions and some reordering:

$$t < \varepsilon^{\delta^-},\ |f^{\delta^-}(x^{\delta^+}) - y_f^{\delta^-}| \not< \varepsilon_f^{\gamma},\ |g^{\delta^-}(x^{\delta^+}) - y_g^{\delta^-}| \not< \varepsilon_g^{\gamma},\ \ldots$$

where $t < \varepsilon^{\delta^-}$ actually reads (with some added wave-front annotation to be used in § 4.8)

$$|\ \boxed{(\underline{f^{\delta^-}(x^{\delta^+})} + \underline{g^{\delta^-}(x^{\delta^+})})} - \boxed{(\underline{y_f^{\delta^-}} + \underline{y_g^{\delta^-}})}\ |\ <\ \lfloor \varepsilon^{\delta^-} \rfloor$$

Now the essential idea of the whole proof is to apply the lemma (5) of Figure 2 via $\{z_0^{\delta^-} \mapsto f^{\delta^-}(x^{\delta^+}),\ z_1^{\delta^-} \mapsto g^{\delta^-}(x^{\delta^+}),\ z_2^{\delta^-} \mapsto y_f^{\delta^-},\ z_3^{\delta^-} \mapsto y_g^{\delta^-}\}$, by which we get:

$$(1^5.3.1^2.2.1.5.1):\quad \boxed{\boxed{t \not\leq |f^{\delta^-}(x^{\delta^+}) - y_f^{\delta^-}| + |g^{\delta^-}(x^{\delta^+}) - y_g^{\delta^-}|}},\ \boxed{t < \varepsilon^{\delta^-}},$$
$$|f^{\delta^-}(x^{\delta^+}) - y_f^{\delta^-}| \not< \varepsilon_f^{\gamma},\ |g^{\delta^-}(x^{\delta^+}) - y_g^{\delta^-}| \not< \varepsilon_g^{\gamma},\ \ldots$$

4.8 Automatic Clean-Up

The rest of the proof is perfectly within the scope of automatic proof search today. When we apply the other transitivity lemma (6) of Figure 2 to $(1^5.3.1^2.2.1.5.1)$ as indicated by the single and double boxes in the goal and the lemma, via $\{\ z_4^{\delta^-} \mapsto t,\ z_6^{\delta^-} \mapsto \varepsilon^{\delta^-},\ z_5^{\delta^-} \mapsto |f^{\delta^-}(x^{\delta^+}) - y_f^{\delta^-}| + |g^{\delta^-}(x^{\delta^+}) - y_g^{\delta^-}|\ \}$, we get:

$$(1^5.3.1^2.2.1.5.1^2):\quad |f^{\delta^-}(x^{\delta^+}) - y_f^{\delta^-}| + |g^{\delta^-}(x^{\delta^+}) - y_g^{\delta^-}| < \varepsilon^{\delta^-},$$
$$\boxed{|f^{\delta^-}(x^{\delta^+}) - y_f^{\delta^-}| \not< \varepsilon_f^{\gamma}},\ \boxed{\boxed{|g^{\delta^-}(x^{\delta^+}) - y_g^{\delta^-}| \not< \varepsilon_g^{\gamma}}},\ \ldots$$

In [44] even the step from $(1^5.3.1^2.2.1.5)$ to $(1^5.3.1^2.2.1.5.1^2)$ is automated with the wave-front annotation of $t < \varepsilon^{\delta^-}$ as given in § 4.7 (which is generated by the givens of $|f^{\delta^-}(x^{\delta^+}) - y_f^{\delta^-}| < \varepsilon_f^{\gamma}$ and $|g^{\delta^-}(x^{\delta^+}) - y_g^{\delta^-}| < \varepsilon_g^{\gamma}$ in the context of $t < \varepsilon^{\delta^-}$ in $(1^5.3.1^2.2.1.5)$), provided that the following lemmas (annotated as wave-rules) are in the rippling system:

$$\boxed{(\underline{z_0^{\delta^-}} + \underline{z_1^{\delta^-}})} - \boxed{(\underline{z_2^{\delta^-}} + \underline{z_3^{\delta^-}})}\ =\ \boxed{\underline{(z_0^{\delta^-} - z_2^{\delta^-})} + \underline{(z_1^{\delta^-} - z_3^{\delta^-})}}$$
$$|\ \boxed{\underline{z_4^{\delta^-}} + \underline{z_5^{\delta^-}}}\ | < z_6^{\delta^-},\quad \boxed{\underline{|z_4^{\delta^-}|} + \underline{|z_5^{\delta^-}|}} \not< z_6^{\delta^-}$$

Applying lemma (7) of Figure 2 (monotonicity of $+$) in the obvious way, we get:

$$(1^5.3.1^2.2.1.5.1^3):\quad |f^{\delta^-}(x^{\delta^+}) - y_f^{\delta^-}| + |g^{\delta^-}(x^{\delta^+}) - y_g^{\delta^-}| \not< \varepsilon_f^{\gamma} + \varepsilon_g^{\gamma},$$
$$|f^{\delta^-}(x^{\delta^+}) - y_f^{\delta^-}| + |g^{\delta^-}(x^{\delta^+}) - y_g^{\delta^-}| < \varepsilon^{\delta^-},\ \ldots$$

The R-substitution $\{\varepsilon_f^{\gamma} \mapsto \frac{\varepsilon^{\delta^-}}{2},\ \varepsilon_g^{\gamma} \mapsto \frac{\varepsilon^{\delta^-}}{2}\}$ closes the remaining open branches of $(1^5.3.1^2.2.1.5.1^3)$ and $(1^5.[1\text{–}2])$ with the lemmas $(3), (8)$ and (9), respectively. The final variable-condition is acyclic indeed. Its graph is depicted in Figure 5 below. The whole proof tree with a minor permutation of the critical β-step is depicted in Figure 7 in § 6.2.

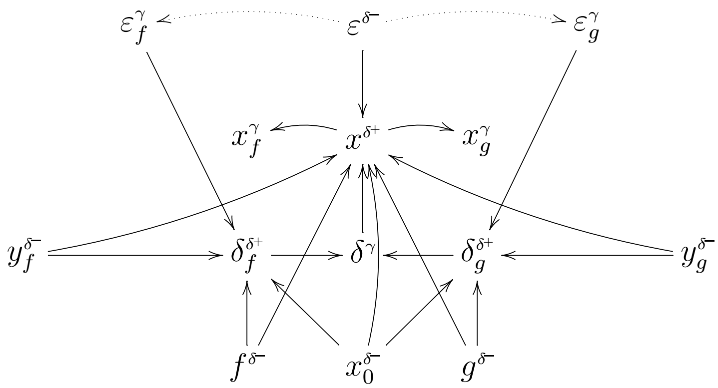

Figure 5: (Acyclic) Variable-Condition R.
With dotted edges: Final State in § 4.8.
Without dotted edges:
State after application of σ, both in § 4.4 and in § 4.6

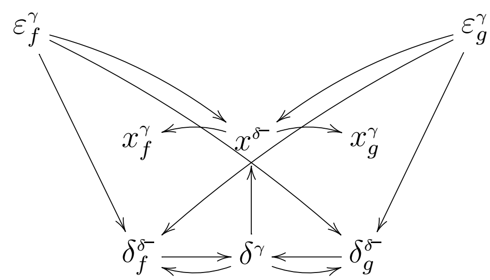

Figure 6: (Cyclic) State of variable-condition R
for alternative proof of § 5.2 with δ^--rules only

5 Discussion

Now that the non-permutability of β at $(1^5.3.1)$ and δ^+ at $(1^5.3.1.2)$ (cf. Figure 3) as well as the non-permutability of β at $(1^5.3.1^2)$ and β at $(1^5.3.1^2.2)$ (cf. Figure 4) have become practically evident by the proof of $(\lim+)$ in § 4, we may ask: *Why did the co-lecturer not believe in what he saw?*

He knew that the only problem with the sequencing of β-steps that occurs either with the δ^--rules or else with the δ^{+^+}-rules [9] is that a bad choice makes the proofs suffer from the repetition of common sub-proofs, which is an optimization problem not subsumed under the notion of non-permutability, cf. § 2.

Thus, we have to make it even clearer why the δ^+-rules are so much in conflict with the β-steps.

5.1 Non-Permutability of β and β is only a Secondary Problem

Notice that the non-permutability of β and δ^+ is the primary problem and the only one we have to explain. It causes the non-permutability of β and β we have seen in Figure 4 as a secondary problem: Indeed, the 2nd β-step in Figure 4 must come before the 1st β-step simply because the 2nd β-step generates the principal δ-formula of the $\delta_0^+(x^{\delta^+})$-step resulting in the rightmost leaf, and this $\delta_0^+(x^{\delta^+})$-step must come before the 1st β-step; namely for the leftmost leaf's first formula $x_f^\gamma \neq x_0^{\delta^-}$ to be of any use in the proof. This means that

$$2^{\text{nd}}\beta \quad <_{\text{superformula}} \quad \delta_0^+(x^{\delta^+}) \quad <_{\beta\text{-}\delta^+\text{-non-permutability}} \quad 1^{\text{st}}\beta$$

causes the non-permutability of 1stβ and 2ndβ by transitivity.

5.2 δ^- instead of δ^+

Let us see how the proof of $(\lim+)$ would look like with the δ^--rules as the only δ-rules available. Roughly speaking, in the proof of § 4, we have to replace each free δ^+-variable $v_n^{\delta^+}$ with a free δ^--variable $v_n^{\delta^-}$ and check how the variable-condition changes: $\delta_0^-(\delta_f^{\delta^-})$ and $\delta_0^-(\delta_g^{\delta^-})$ applied to $(1^5.3)$ of § 4.2 and $(1^5.3.1.2)$ of § 4.3 (cf. Figure 3) add $\{\varepsilon_f^\gamma, \varepsilon_g^\gamma, \delta^\gamma\} \times \{\delta_f^{\delta^-}\}$ and $\{\varepsilon_f^\gamma, \varepsilon_g^\gamma, \delta^\gamma\} \times \{\delta_g^{\delta^-}\}$ to the initially empty variable-condition R, respectively. $\delta_0^-(x^{\delta^-})$ applied roughly at $(1^5.3.1.2.1)$ adds $\{\varepsilon_f^\gamma, \varepsilon_g^\gamma, \delta^\gamma\} \times \{x^{\delta^-}\}$ later.

Thus, after applying

$$\sigma^- := \{x_f^\gamma \mapsto x^{\delta^-},\ x_g^\gamma \mapsto x^{\delta^-},\ \delta^\gamma \mapsto \min(\delta_f^{\delta^-}, \delta_g^{\delta^-})\}$$

the σ^--updated variable-condition is extended by

$$\{(x^{\delta^-}, x_f^\gamma), (x^{\delta^-}, x_g^\gamma), (\delta_f^{\delta^-}, \delta^\gamma), (\delta_g^{\delta^-}, \delta^\gamma)\}$$

and looks as in Figure 6 above. Compared to the graph of Figure 5, it is small but cyclic: Among others, the two curved edges at the very bottom are new and cause the cycles. Thus, σ^- is no R-substitution at all and cannot be applied.

Therefore, in our example proof of § 4 as depicted in Figure 3, we have to move the γ-step applied to (1^3) down below $(1^5.3.1.2.1)$. Note that we cannot move it deeper because it has to preceed the step $\delta_0^-(x^{\delta^-})$: Indeed, the principal formula of this δ^--step is a subformula of the side formula of the γ-step. A fortiori, this movement of the γ-step applied to (1^3) forces the problematic β-step at $(1^5.3.1)$ to be moved below $(1^5.3.1.2.1)$, too; simply because its principal β-formula is the side formula of the γ-step.

Indeed, if we replace the δ^+-rules with δ^--rules, the non-permutability of the β- and the δ^+-steps is hidden behind the well-known non-permutability of the γ- and the δ^--steps, cf. § 2. Only when the latter non-permutability is removed by replacing the δ^--rules with δ^+-rules, the former becomes visible.

5.3 Free δ^+-Variables can Escape their Quantifiers' Scopes

The non-permutability of the β- and δ^+-steps is closely related to the following strange aspect of the δ^+-rules, which they share with the δ^{+^+}-rules [9], the δ^*-rules [7], and the δ^{*^*}-rules [11], but not with the δ^ε-rules [16] and the δ^--rules. While soundness of both the δ^-- and δ^+-rules and preservation of solutions of the δ^--rules are immediate, the preservation of solutions of the δ^+-rules requires the restriction of the values of the free δ^+-variables by choice-conditions [42, Theorem 2.49]. Although there is no space here for introducing the semantics of the several kinds of free variables of [42], the reader may grasp the idea of the following example, namely that a solution for x^γ that makes the lower sequent true, may make the upper sequent false:

Example 5.1 (Reduction & Liberalized δ, [42, Example 2.29])
In [42, Example 2.8], a δ^+-step reduces $\quad \forall y.\ \neg\mathsf{P}(y), \quad \mathsf{P}(x^\gamma), \ \ldots$
to $\quad \neg\mathsf{P}(y^{\delta^+}), \quad \mathsf{P}(x^\gamma), \ \ldots$
with the empty variable-condition $R := \emptyset$.

Let us first argue semantically: The lower sequent is $(e, \mathcal{S})$-valid for the $(\mathcal{S}, R)$-valuation e given by

$$e(x^\gamma)(\delta) := \delta(y^{\delta^+}),$$

which sets the value of x^γ to the value of y^{δ^+}. The upper sequent, however, is not $(e, \mathcal{S})$-valid when $\mathsf{P}^{\mathcal{S}}(a)$ is TRUE and $\mathsf{P}^{\mathcal{S}}(b)$ is FALSE for some a, b from the universe of the structure $\mathcal{S}$. To see this, take some valuation δ with $\delta(y^{\delta^+}) := b$. Then x^γ and y^{δ^+} both evaluate to b, the lower sequent to $\mathsf{TRUE}, \ \mathsf{FALSE}$, and the upper sequent to $\mathsf{FALSE}, \ \mathsf{FALSE}$.

No matter whether this semantical argumentation can become clear here, the following syntactical variant will do similarly well: After applying the R-substitution

$$\mu^+ := \{x^\gamma \mapsto y^{\delta^+}\},$$

the lower sequent is a tautology, whereas the upper sequent is not.

This cannot happen with the δ^--rules: Their application instead of the δ^+-rules adds $\{(x^\gamma, y^{\delta^-})\}$ to the variable-condition, thereby blocking

$$\mu^- := \{x^\gamma \mapsto y^{\delta^-}\},$$

simply because μ^- is no $\{(x^\gamma, y^{\delta^-})\}$-substitution, cf. Definition 3.1.

From a semantical point of view, however, the e displayed above is no $(\mathcal{S}, R)$-valuation for the extended variable-condition anymore.

Roughly speaking, via μ^+, the δ^+-variable y^{δ^+} escapes the scope of the quantifier $\forall y$ on the bound variable y which was eliminated by the introduction of y^{δ^+}. At least with matrix calculi and indexed formulas trees [2, 37], this "escaping" is a natural way to talk about this strange liberality of the δ^+-rule. And it also happens in Figure 3 of the proof of $(\lim+)$: Taking the tree of Figure 3 to be an indexed formula tree, roughly speaking, the quantifier for $\delta_g^{\delta^+}$ is situated at the term position $(1^5.3.1.2)$, but, via σ, it escapes to term position $(1^5.3.1.1)$.

5.4 $\delta^{+^{+}}$ instead of δ^{+}

Let us see how the proof of $(\lim+)$ would look like with the $\delta^{+^{+}}$-rules [9] as the only δ-rules available. This does not change anything in the proof as given in § 4, but allows us to use the identical free δ^{+}-variable $\delta_{g}^{\delta^{+}}$ again when repeating the δ-step which introduced it. Thus, starting from $(1^{5}.3.1.1)$ of § 4.3, we can repeat some of the steps done in proof of $(1^{5}.3.1.2)$, namely " $\delta_{0}^{+}(\delta_{g}^{\delta^{+}}),\ \alpha_{0}^{2}$ " of Figure 3, but now as " $\delta_{0}^{+^{+}}(\delta_{g}^{\delta^{+}}),\ \alpha_{0}^{2}$ ". Note that the δ^{+}-rules would allow $\delta_{0}^{+}(\delta_{G}^{\delta^{+}})$ only, with new $\delta_{G}^{\delta^{+}}$. The resulting sequent is

$$(1^{5}.3.1.1.1):\quad 0<\min(\delta_{f}^{\delta^{+}},\delta_{g}^{\delta^{+}}),\ \varOmega$$

It is like $(1^{5}.3.1.2.1)$ of § 4.3, but with the β_{2}-side formula of the critical β-step replaced with the β_{1}-side formula $0<\min(\delta_{f}^{\delta^{+}},\delta_{g}^{\delta^{+}})$. This formula admits to close this branch with the formulas $0\not<\delta_{f}^{\delta^{+}}$ and $0\not<\delta_{g}^{\delta^{+}}$ (as listed in $\varOmega$ of Figure 2), applying lemma (4) of Figure 2 as at the end of § 4.6.

Notice that this proof with the $\delta^{+^{+}}$-rules does not have a higher number of γ-steps than the proof attempt failing in § 4.5. Also the maximum number of δ-steps per formula and *per path* is still 1. Nevertheless, the multiple expansion of the same δ-formula in different paths is somehow counter-intuitive and nothing a working mathematician would expect. In indexed formula trees based on the $\delta^{+^{+}}$-rules, all δ-formulas are treated only once. This again means that these matrix versions are more human-oriented than the tableau or sequent versions.

6 Proof of the Non-Permutability of β and δ^+

As we have seen in § 5.2, the non-permutable β-step necessarily follows a γ-step that would be non-permutable without the liberalization from δ^- to δ^+. It follows indeed *necessarily*, because the principal formula of the β-step is the side formula of the γ-step. Although

- the γ-step $\gamma_0(\min(\delta_f^{\delta^+}, \delta_g^{\delta^+}))$ is permutable with the liberalized δ^+-step $\delta_0^+(\delta_g^{\delta^+})$,
- the γ-step $\gamma_0(\min(\delta_f^{\delta^-}, \delta_g^{\delta^-}))$, however, is non-permutable with the δ^--step $\delta_0^-(\delta_g^{\delta^-})$,

and even with the liberalization

- the β-step is still non-permutable with the δ^+-step $\delta_0^+(\delta_g^{\delta^+})$.

As the principal formula of the β-step can be regenerated by a second expansion of the principal formula of the γ-step, we cannot prove the non-permutability unless we restrict the γ-multiplicity. But, according to the description of the notion of non-permutability in § 2, we may indeed restrict the γ-multiplicity, in which case the crucial step, namely Lemma 4.1, admits the following semantical proof.

6.1 Proof of Lemma 4.1 at the end of § 4.5

Let us remove the three γ-formulas which form the sequent Γ (cf. Figure 2) from the sequents $(1^5.1)$, $(1^5.2)$ (cf. § 4.2), and $(1^5.3.1.1)$ (cf. § 4.3). As these γ-formulas were already once expanded at (1^3) and (1^4) (cf. Figure 3), this removal represents a restriction of the γ-multiplicity of the removed γ-formulas to 1, and results in the following sequents (after some reordering):

$$(1^5.1\backslash\Gamma+):\quad 0<\varepsilon_f^\gamma,\ 0\not<\varepsilon^{\delta^-},$$
$$\neg\left(0<\varepsilon_g^\gamma \Rightarrow \exists\delta_g{>}0.\ \forall x_g{\neq}x_0^{\delta^-}.\left(\begin{array}{rl} & |g^{\delta^-}(x_g)-y_g^{\delta^-}| < \varepsilon_g^\gamma \\ \Leftarrow & |x_g-x_0^{\delta^-}| < \delta_g\end{array}\right)\right),$$
$$0<\min(\delta_f^{\delta^+}, \delta_g^{\delta^+}) \wedge \forall x{\neq}x_0^{\delta^-}.\left(\begin{array}{rl} & |(f^{\delta^-}(x)+g^{\delta^-}(x))-(y_f^{\delta^-}+y_g^{\delta^-})| < \varepsilon^{\delta^-} \\ \Leftarrow & |x-x_0^{\delta^-}| < \min(\delta_f^{\delta^+}, \delta_g^{\delta^+})\end{array}\right)$$

$$(1^5.2\backslash\Gamma+):\quad 0<\varepsilon_g^\gamma,\ 0\not<\varepsilon^{\delta^-},$$
$$\neg\exists\delta_f{>}0.\ \forall x_f{\neq}x_0^{\delta^-}.\left(\begin{array}{rl} & |f^{\delta^-}(x_f)-y_f^{\delta^-}| < \varepsilon_f^\gamma \\ \Leftarrow & |x_f-x_0^{\delta^-}| < \delta_f\end{array}\right),$$
$$0<\min(\delta_f^{\delta^+}, \delta_g^{\delta^+}) \wedge \forall x{\neq}x_0^{\delta^-}.\left(\begin{array}{rl} & |(f^{\delta^-}(x)+g^{\delta^-}(x))-(y_f^{\delta^-}+y_g^{\delta^-})| < \varepsilon^{\delta^-} \\ \Leftarrow & |x-x_0^{\delta^-}| < \min(\delta_f^{\delta^+}, \delta_g^{\delta^+})\end{array}\right)$$

$$(1^5.3.1.1\backslash\Gamma+):\quad 0<\min(\delta_f^{\delta^+}, \delta_g^{\delta^+}),\ 0\not<\varepsilon^{\delta^-},$$
$$\neg\left(0<\delta_f^{\delta^+}\ \wedge\ \forall x_f{\neq}x_0^{\delta^-}.\left(\begin{array}{rl} & |f^{\delta^-}(x_f)-y_f^{\delta^-}| < \varepsilon_f^\gamma \\ \Leftarrow & |x_f-x_0^{\delta^-}| < \delta_f^{\delta^+}\end{array}\right)\right),$$
$$\neg\exists\delta_g.\left(0<\delta_g\ \wedge\ \forall x_g{\neq}x_0^{\delta^-}.\left(\begin{array}{rl} & |g^{\delta^-}(x_g)-y_g^{\delta^-}| < \varepsilon_g^\gamma \\ \Leftarrow & |x_g-x_0^{\delta^-}| < \delta_g\end{array}\right)\right)$$

The related variable-condition R is shown in Figure 5 (without the dotted edges) and the current R-choice-condition C is given as

$$\left\{\begin{array}{l}
x^{\delta^+} \mapsto \neg\left(x^{\delta^+}{\neq}x_0^{\delta^-} \Rightarrow \left(\begin{array}{ll} & |(f^{\delta^-}(x^{\delta^+}){+}g^{\delta^-}(x^{\delta^+})){-}(y_f^{\delta^-}{+}y_g^{\delta^-})| < \varepsilon^{\delta^-} \\ \Leftarrow & |x^{\delta^+}{-}x_0^{\delta^-}| < \min(\delta_f^{\delta^+}, \delta_g^{\delta^+}) \end{array}\right)\right), \\
\delta_f^{\delta^+} \mapsto \left(0{<}\delta_f^{\delta^+} \wedge \forall x_f{\neq}x_0^{\delta^-}. \left(\begin{array}{ll} & |f^{\delta^-}(x_f){-}y_f^{\delta^-}| < \varepsilon_f^{\gamma} \\ \Leftarrow & |x_f{-}x_0^{\delta^-}| < \delta_f^{\delta^+} \end{array}\right)\right), \\
\delta_g^{\delta^+} \mapsto \left(0{<}\delta_g^{\delta^+} \wedge \forall x_g{\neq}x_0^{\delta^-}. \left(\begin{array}{ll} & |g^{\delta^-}(x_g){-}y_g^{\delta^-}| < \varepsilon_g^{\gamma} \\ \Leftarrow & |x_g{-}x_0^{\delta^-}| < \delta_g^{\delta^+} \end{array}\right)\right)
\end{array}\right\}$$

It now suffices to show that there is no proof of $(1^5.1\backslash\Gamma+)$, $(1^5.2\backslash\Gamma+)$, and $(1^5.3.1.1\backslash\Gamma+)$ with the δ^-- and δ^+-rules as the only δ-rules available.

We do this with a trivial transformation given by the substitution

$$\nu := \{\delta_f^{\delta^+}{\mapsto}\delta_f^{\delta^-},\ \delta_g^{\delta^+}{\mapsto}\delta_g^{\delta^-}\}$$

of an assumed proof of $(1^5.1\backslash\Gamma+)$, $(1^5.2\backslash\Gamma+)$, and $(1^5.3.1.1\backslash\Gamma+)$ on the one hand, and with a deviation over invalidity and soundness on the other hand, as follows:

Instantiating the sequents $(1^5.1\backslash\Gamma+)$, $(1^5.2\backslash\Gamma+)$, and $(1^5.3.1.1\backslash\Gamma+)$ by ν we get the sequents

$$\begin{array}{l}
(1^5.1\backslash\Gamma-)\text{:}\ \ 0{<}\varepsilon_f^{\gamma},\ 0{\not<}\varepsilon^{\delta^-}, \\
\qquad \neg\left(0{<}\varepsilon_g^{\gamma} \Rightarrow \exists\delta_g{>}0.\ \forall x_g{\neq}x_0^{\delta^-}. \left(\begin{array}{ll} & |g^{\delta^-}(x_g){-}y_g^{\delta^-}| < \varepsilon_g^{\gamma} \\ \Leftarrow & |x_g{-}x_0^{\delta^-}| < \delta_g \end{array}\right)\right), \\
\qquad 0{<}\min(\delta_f^{\delta^-}, \delta_g^{\delta^-}) \wedge \forall x{\neq}x_0^{\delta^-}. \left(\begin{array}{ll} & |(f^{\delta^-}(x){+}g^{\delta^-}(x)){-}(y_f^{\delta^-}{+}y_g^{\delta^-})| < \varepsilon^{\delta^-} \\ \Leftarrow & |x{-}x_0^{\delta^-}| < \min(\delta_f^{\delta^-}, \delta_g^{\delta^-}) \end{array}\right)
\end{array}$$

$$\begin{array}{l}
(1^5.2\backslash\Gamma-)\text{:}\ \ 0{<}\varepsilon_g^{\gamma},\ 0{\not<}\varepsilon^{\delta^-},\ \neg\exists\delta_f{>}0.\ \forall x_f{\neq}x_0^{\delta^-}. \left(\begin{array}{ll} & |f^{\delta^-}(x_f){-}y_f^{\delta^-}| < \varepsilon_f^{\gamma} \\ \Leftarrow & |x_f{-}x_0^{\delta^-}| < \delta_f \end{array}\right), \\
\qquad 0{<}\min(\delta_f^{\delta^-}, \delta_g^{\delta^-}) \wedge \forall x{\neq}x_0^{\delta^-}. \left(\begin{array}{ll} & |(f^{\delta^-}(x){+}g^{\delta^-}(x)){-}(y_f^{\delta^-}{+}y_g^{\delta^-})| < \varepsilon^{\delta^-} \\ \Leftarrow & |x{-}x_0^{\delta^-}| < \min(\delta_f^{\delta^-}, \delta_g^{\delta^-}) \end{array}\right)
\end{array}$$

$$\begin{array}{l}
(1^5.3.1.1\backslash\Gamma-)\text{:} \qquad 0{<}\min(\delta_f^{\delta^-}, \delta_g^{\delta^-}),\ 0{\not<}\varepsilon^{\delta^-}, \\
\qquad \neg\left(\ 0{<}\delta_f^{\delta^-}\ \wedge\ \forall x_f{\neq}x_0^{\delta^-}.\ \left(\ |f^{\delta^-}(x_f){-}y_f^{\delta^-}| < \varepsilon_f^{\gamma}\ \Leftarrow\ |x_f{-}x_0^{\delta^-}| < \delta_f^{\delta^-}\ \right)\ \right), \\
\qquad \neg\exists\delta_g.\ \left(0{<}\delta_g\ \wedge\ \forall x_g{\neq}x_0^{\delta^-}. \left(\begin{array}{ll} & |g^{\delta^-}(x_g){-}y_g^{\delta^-}| < \varepsilon_g^{\gamma} \\ \Leftarrow & |x_g{-}x_0^{\delta^-}| < \delta_g \end{array}\right)\right)
\end{array}$$

The conjunction of these sequents is invalid according to the standard semantics for parameters as well as the semantics of [42]. This can be seen by

$$\{\ \delta_f^{\delta^-}{\mapsto}1,\quad \delta_g^{\delta^-}{\mapsto}0,\quad \varepsilon^{\delta^-}{\mapsto}1,\quad x_0^{\delta^-}{\mapsto}0,\quad y_f^{\delta^-}{\mapsto}0,\quad y_g^{\delta^-}{\mapsto}0,\quad f^{\delta^-} \mapsto \lambda x.0,\quad g^{\delta^-} \mapsto \lambda x.0\ \}.$$

Indeed, if we instantiate $(1^5.1\backslash\Gamma-)$, $(1^5.2\backslash\Gamma-)$, and $(1^5.3.1.1\backslash\Gamma-)$ with this substitution and then $\lambda\beta$-normalize and simplify these sequents by equivalence transformations in the model of the real numbers $\mathbf{R}$, we get the three sequents

$$0{<}\varepsilon_f^{\gamma},\ \mathsf{false},\ \neg\left(0{<}\varepsilon_g^{\gamma} \Rightarrow \left(\begin{array}{ll} & 0{<}\varepsilon_g^{\gamma} \\ \Leftarrow & \forall\delta_g{>}0.\ \exists x_g{\neq}0.\ |x_g|{<}\delta_g \end{array}\right)\right),\ \mathsf{false}$$

$$0{<}\varepsilon_g^\gamma,\ \mathsf{false},\ \neg\left(\begin{array}{ll} & 0{<}\varepsilon_f^\gamma \\ \Leftarrow & \forall\delta_f{>}0.\ \exists x_f{\neq}0.\ |x_f|{<}\delta_f \end{array}\right),\ \mathsf{false}$$

$$\mathsf{false},\ \mathsf{false},\ \neg(0{<}\varepsilon_f^\gamma \Leftarrow \exists x_f{\neq}0.\ |x_f|{<}1),\ \neg\left(\begin{array}{ll} & 0{<}\varepsilon_g^\gamma \\ \Leftarrow & \forall\delta_g{>}0.\ \exists x_g{\neq}0.\ |x_g|{<}\delta_g \end{array}\right)$$

Further equivalence transformation in $\mathbf{R}$ results in the three contradictory sequents

$$0{<}\varepsilon_f^\gamma$$

$$0{<}\varepsilon_g^\gamma,\ 0{\not<}\varepsilon_f^\gamma$$

$$0{\not<}\varepsilon_f^\gamma,\ 0{\not<}\varepsilon_g^\gamma$$

Thus, as our calculus is sound, it cannot prove $(1^5.1\backslash\Gamma-)$, $(1^5.2\backslash\Gamma-)$, and $(1^5.3.1.1\backslash\Gamma-)$ in parallel.

As the δ^+-rules treat free δ^-- and free δ^+-variables alike, and as the δ^--rules generate a smaller variable-condition for free δ^-- instead of free δ^+-variables in the principal sequents (cf. $\mathcal{V}_{\gamma\delta^+}(\ldots)$ in Figure 1), a proof of $(1^5.1\backslash\Gamma+)$, $(1^5.2\backslash\Gamma+)$, and $(1^5.3.1.1\backslash\Gamma+)$ would immediately translate into a proof of $(1^5.1\backslash\Gamma-)$, $(1^5.2\backslash\Gamma-)$, and $(1^5.3.1.1\backslash\Gamma-)$ with unchanged inference rules, just by application of the substitution ν.

Thus, we conclude that there is no proof of $(1^5.1\backslash\Gamma+)$, $(1^5.2\backslash\Gamma+)$, and $(1^5.3.1.1\backslash\Gamma+)$. q.e.d.

Note that the above trivial proof transformation does not result in a sound proof if we replace the δ^+-rules with the δ^{+^+}-rules: Indeed, the δ^{+^+}-rules may re-use $\delta_g^{\delta^+}$, but not $\delta_g^{\delta^-}$.

6.2 Defining Permutability

A reader with a good mathematical intuition can and should directly consider the non-permutability of β- and δ^+-steps as a corollary of Lemma 4.1 proved above. A formalist, however, may well require some rigorous definition of permutability. There were good reasons not to present a formal definition of permutability earlier in this paper:

1. The logically weakest reasonable definitions of permutability I can think of, still result in the non-permutability we want to show. Indeed, we may choose any definition of permutability that contradicts Lemma 4.1. For instance, as it strengthens our non-permutability result, we should (and will) use a notion that is weaker than the following standard one: Two inference steps S_1 and S_0 are *locally directly permutable* if replacing an occurrence of $\frac{S_0}{S_l \quad S_1 \quad S_r}$ in a closed proof tree (where S_1 is also applicable instead of S_0) with $\frac{S_1}{\frac{S_0}{S_l} \quad S_0 \quad \frac{S_0}{S_r}}$ results—*mutatis mutandis*—in a closed proof tree.

2. From the viewpoint of philosophy of mathematics it is bad practice to become too concrete with intuitively clear notions. For example, we should not say precisely which set theory we use on the meta-level as long as Zermelo–Fraenkel, Neumann–Bernays–Gödel, Quine's NF, Quine's ML, Tarski–Grothendieck and non-wellfounded set theories [1, 8] &c. all satisfy our needs. Although the case of permutability is not as self-evident as the case of set theory, the low rigor of our notion of permutability was sufficient until now. Indeed, there is no definition of permutability or non-permutability in Wallen's whole book [37], although the avoidance of non-permutability is one of its main subjects, cf. § 2.

3. My formalization of the notion of permutability depends on the notions of a *principal meta-variable* of an *inference rule* and is somewhat technical and difficult, even in the rudimental form we will present below.

To avoid clutter, we define permutability only for sequent calculi. The definition for tableau calculi is analogous. Formally, for each inference rule, we have to define which meta-variables are principal and which are not. On the one hand, the meta-variables of the principal formulas have to be principal, and an instantiation of all principal meta-variables must determine the existence of an instantiation of the other meta-variables such that the inference rule becomes applicable. On the other hand, it is not appropriate to define all meta-variables of an inference rule to be principal, because this results in a general non-permutability of inference steps.

Definition 6.1 (Principal Meta-Variables)
In our inference rules of Figure 1 in § 3.1 exactly the meta-variables A, B, x, t, x^{δ^-}, and x^{δ^+} are *principal*; and the other meta-variables, i.e. Γ, Π, are not principal. In lemma application steps as explained in § 3.2, the A_k and C_i are principal, whereas the B_j are not. For technical simplicity, we ignore our definitional expansion steps on $\forall$, $\exists$, $\lim$, assuming a complete expansion at the calculus level.

Definition 6.2 (Inference Step)

A *proof tree* is a labeled tree whose root is labeled with a sequent and whose paths are labeled with sequents and inference steps alternately, such that there is a proof history of applicable inference steps (expansion steps) and global applications of R-substitutions on free γ-variables (which instantiate the free γ-variables of their domains in all occurrences in all labels of the proof tree, i.e. in all sequents *and in all inference steps*), starting from a proof tree consisting only of a root node. (Of course, the parent and child nodes of a node labeled with an inference step must be labeled with the conclusion and the premises of this inference step, respectively.)

A proof tree is *closed* if all its leaves that are not labeled with inference steps are labeled with axioms.

An *inference step* is a triple (I, π, ϱ) labeling a node in a proof tree where I is an inference rule and π and ϱ are substitutions of the principal and non-principal meta-variables of I, respectively; so that $I(\pi \uplus \varrho)$ describes the inference step with parent (conclusion) and child (premise) nodes as an instance of the inference rule I.

Note that in Definition 6.2 we indeed have to refer to the proof history because the δ^+-step $\delta_0^+(\delta_g^{\delta^+})$ applied to $(1^5.3.1)$ at the beginning of § 4.6 would not be admitted if we applied the R-substitution σ before expanding the proof tree by the δ^+-step. This is because δ^+-steps have to introduce *new* free δ-variables, and σ would already introduce $\delta_g^{\delta^+}$ before.

Roughly speaking, permutability of two steps S_1 and S_0 simply means the following: *In a closed proof tree where S_0 precedes S_1 and where S_1 was already applicable before S_0, we can do the step S_1 before S_0 and find a closed proof tree nevertheless.*

Definition 6.3 (Permutability)

Let (I_1, π_1, ϱ_1) and (I_0, π_0, ϱ_0) be two inference steps.

(I_1, π_1, ϱ_1) and (I_0, π_0, ϱ_0) are *permutable for a given threshold m for γ-multiplicity* if

for any closed proof tree T with γ-multiplicity m satisfying that

1. n_i is an inference node in T labeled with (I_i, π_i, ϱ_i), for $i \in \{0, 1\}$,
2. n_0, n_1 are, in this order and with only a sequent node in between, on the same path in T from the root to a leaf, and
3. there is a substitution ϕ such that the parent sequents (conclusions) of $I_0(\pi_0 \uplus \varrho_0)$ and of $I_1(\pi_1 \uplus \phi)$ are identical;

there is a closed proof tree with γ-multiplicity m which differs from T only in the subtree starting with n_0 and the root label of this subtree is (I_1, π_1, ϕ).

(I_1, π_1, ϱ_1) and (I_0, π_0, ϱ_0) are *permutable* if they are permutable for any given threshold $m \in \mathbf{N}$ of γ-multiplicity.

I_1 and I_0 are *generally permutable* if all inference steps of the forms (I_1, π_1, ϱ_1) and (I_0, π_0, ϱ_0) are permutable.

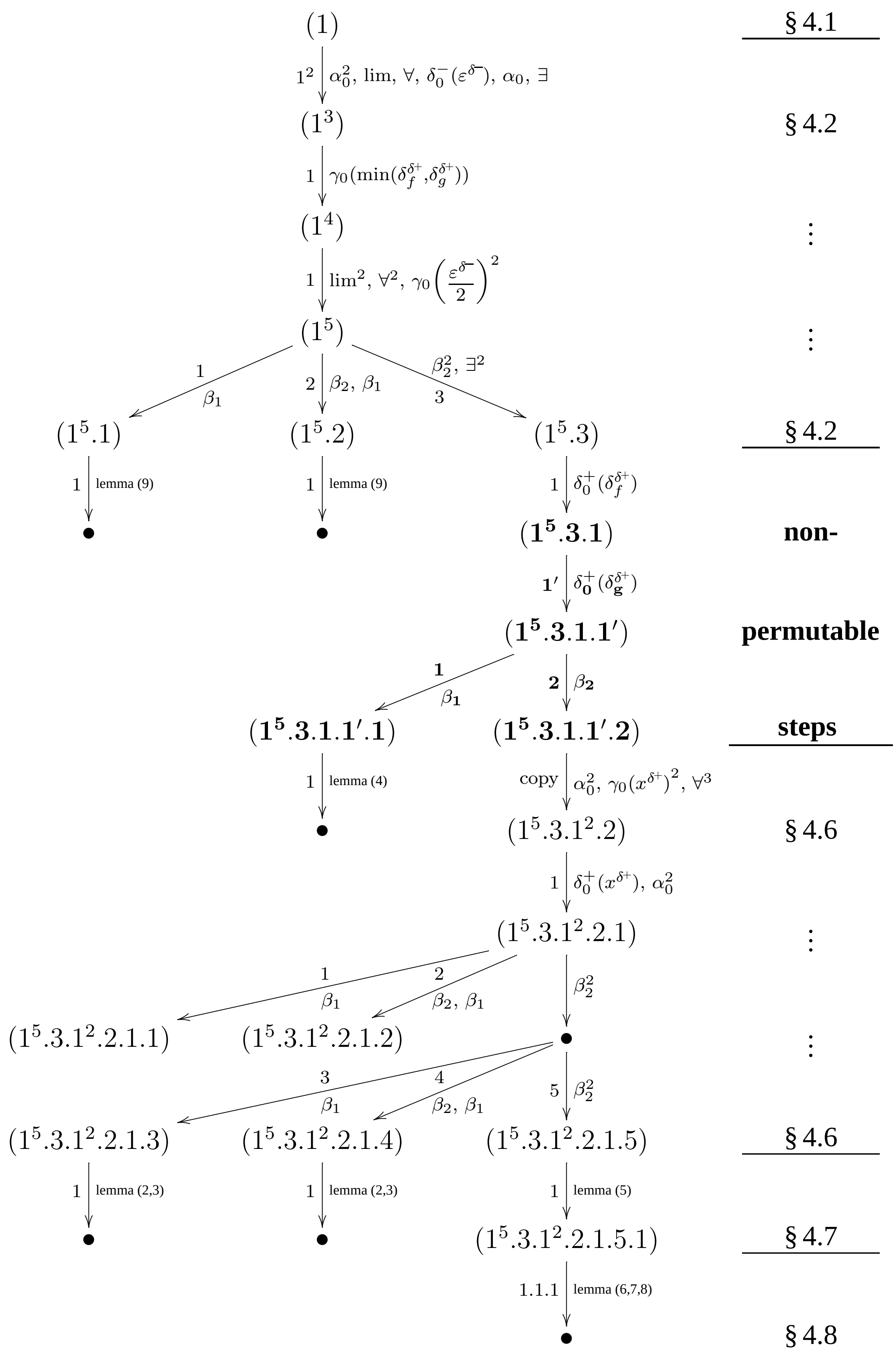

Figure 7: Closed proof tree with non-permutable β and δ^+-step

Example 6.4
For inferring the non-permutability of β and δ^{+} from Lemma 4.1, we have to instantiate Definition 6.3 as follows:

$$
\begin{array}{l}
n_0 \ \approx\ (1^5.3.1){\longrightarrow}(1^5.3.1^2)\ \text{(cf. § 4.6)}\\
I_0\ \text{ is }\ (\delta^{+},\neg\exists)\ \text{of Figure 1 in § 3.1}\\
\pi_0\ =\ \left\{\begin{array}{lll}
x & \mapsto & \delta_g;\\
x^{\delta^{+}} & \mapsto & \delta_g^{\delta^{+}};\\
A & \mapsto & \left(\ 0{<}\delta_g \wedge \exists x_g{\neq}x_0^{\delta^{-}}.\ \left(\begin{array}{rl} & |g^{\delta^{-}}(x_g){-}y_g^{\delta^{-}}| < \frac{\varepsilon^{\delta^{-}}}{2}\\ \Leftarrow & |x_g{-}x_0^{\delta^{-}}| < \delta_g\end{array}\right)\right)
\end{array}\right\}\\
\varrho_0\ =\ \left\{\begin{array}{l}
\mathit{\Gamma}\ \mapsto\ \left(\begin{array}{l} 0{<}\min(\delta_f^{\delta^{+}},\delta_g^{\delta^{+}})\\ \wedge\ \ \forall x{\neq}x_0^{\delta^{-}}.\ \left(\begin{array}{rl} & \left|\begin{array}{l}(f^{\delta^{-}}(x){+}g^{\delta^{-}}(x))\\ -(y_f^{\delta^{-}}{+}y_g^{\delta^{-}})\end{array}\right| < \varepsilon^{\delta^{-}}\\ \Leftarrow & |x{-}x_0^{\delta^{-}}| < \min(\delta_f^{\delta^{+}},\delta_g^{\delta^{+}})\end{array}\right)\end{array}\right),\ \ldots;\\
\mathit{\Pi}\ \mapsto\ \ldots
\end{array}\right\}
\end{array}
$$

$n_1\ \approx$ "a new step of an alternative closed proof tree that results from the closed proof tree of § 4.6 by permuting the β-step at $(1^5.3.1^2)$ and the steps $\alpha^2, \gamma_0(x^{\delta^{+}})^2$ applied to $(1^5.3.1)$. This alternative proof tree is depicted in Figure 7 above. (For pedagogical reasons only, we delayed the potentially sinful β-step until we were forced to do it.)"

I_1 is $(\beta, \wedge)$ of Figure 1 in § 3.1

$$
\pi_1\ =\ \left\{\begin{array}{lll}
A & \mapsto & 0{<}\min(\delta_f^{\delta^{+}},\delta_g^{\delta^{+}});\\
B & \mapsto & \forall x{\neq}x_0^{\delta^{-}}.\ \left(\begin{array}{rl} & \left|\begin{array}{l}(f^{\delta^{-}}(x){+}g^{\delta^{-}}(x))\\ -(y_f^{\delta^{-}}{+}y_g^{\delta^{-}})\end{array}\right| < \varepsilon^{\delta^{-}}\\ \Leftarrow & |x{-}x_0^{\delta^{-}}| < \min(\delta_f^{\delta^{+}},\delta_g^{\delta^{+}})\end{array}\right)
\end{array}\right\}
$$

Now, the non-permutability of the critical β- and δ^{+}-steps of Example 6.4 follows from Lemma 4.1, because there is no alternative proof tree which differs only in the subtree starting at n_0 and having a new subtree there starting with the critical β-step. The deeper reason for this is that the instantiated free γ-variables occur outside the subtree of the δ^{+}-step, cf. § 5.3. According to Lemma 4.1, there is no proof of $(1^5.1)$, $(1^5.2)$ and $(1^5.3.1.1)$ with the instantiation by σ given by the failed proof attempt. Since the partial instantiation by σ agrees with the full instantiation in the closed proof tree of the successful proof of Figure 7, we have the required witness for the non-permutability of β and δ^{+}, indeed. Thus, as corollaries we get:

Corollary 6.5 *On a threshold for γ-multiplicity of 1, the inference steps*

$$((\beta,\wedge),\pi_1,\varrho_1)\quad \textit{and}\quad ((\delta^{+},\neg\exists),\pi_0,\varrho_0)$$

(as labels of the nodes n_1 and n_0, resp.) as given in Example 6.4 are not permutable.

Theorem 6.6 *β- and δ^{+}-steps are not generally permutable,*

- *neither in the sequent calculus of [42] (cf. our Figure 1 in § 3.1),*
- *nor in standard free-variable tableau calculi with δ^{+}-rules as the only δ-rules, such as the ones in [14, 18].*

7 Conclusion

Even with more liberalized δ-rules available today (such as δ^{+^+}-, δ^*-, δ^{*^*}-, and δ^ε-rules, cf. § 5.3), the δ^+-rules stay important, both conceptually and for stepwise presentation and limitation of complexity in teaching, research, and publication. For instance, the δ^+-rules are the free-variable tableau rules used in the current edition of Fitting's excellent textbook [14]. Moreover, until very recently [12] nobody realized that the δ^*- and δ^{*^*}-rules were unsound in their original publications (incl. their corrigenda!).

When the δ^+-rules occurred first in [18], they seemed so simple and straightforward. Today, a dozen years later, they are still not completely understood. We have shown that the δ^+-rules have unrealized properties yet, such as the non-permutability of β- and δ^+-steps. Indeed, there are several *open problems*, such as, from theoretical to practical:

7.1 Complexity?

Does the non-elementary reduction in proof size [7] from the δ^-- to the δ^{+^+}-rules mean a non-elementary reduction in proof size from δ^- to δ^+, or from δ^+ to δ^{+^+} (exponential at least [9]), or both?

7.2 More Non-Permutabilities?

Why was the non-permutability of β and δ^+ not noticed before? May there be others around?

7.3 Optimization?

Although the non-permutability of β- and δ^+-steps is not visible with non-liberalized δ-rules and not serious in theory with further liberalized δ-rules, it is always present and of major importance in practice; both for efficiency of proof search and for human-oriented proof presentation. The same holds for the optimization problem of finding a good order of application for the β-steps.

7.4 Are the known notions of Completeness relevant in practice?

The mere existence of a proof is not sufficient for mathematics assistance systems, where we need the existence of a proof that closely mirrors the proof the mathematician interacting with the system has in mind, searches for, or plans.

Freshmen who think that the δ^--rules would admit human-oriented proof construction should try to do the proof of $(\lim+)$ with the δ^--rules as the only δ-rules. There will be more reasons and occasions to use the presentation of this complete and interesting example proof for further reference!

I must admit, however, that I do not know how to grasp a practically relevant notion of completeness. The sequent calculus of our inductive theorem prover QUODLIBET [6] has been improved over a dozen years of practical application to admit our proofs; and still needs and gets further improvement.

The automatic generation of a non-trivial proof for a given input conjecture is typically not possible today and probably will never be. Thus, besides some rare exceptions—as the automation of proof search will always fail on the lowest logic level from time to time—the only chance for automatic theorem proving to become useful for mathematicians is a synergetic interplay between the mathematician and the machine. For this interplay—to give the human user a chance to interact—the calculus *itself* must be human-oriented. Indeed, it does not suffice to compute human-oriented representations; not in the end, and—as the syntactical problems have to be presented accurately—also not intermediately in a user interface.

Thus, also the possibility to overcome the non-permutability of β and δ^+ by replacing the δ^+-rules with δ^{+^+}-rules as described in § 5.4 is not adequate for human-oriented reasoning, for which we need matrix calculi and indexed formula trees [2, 37] to admit a lazy sequencing of β-steps, so that the connection-driven path construction may tell us in the end, which sequencing of the β-steps we need.[4]

7.5 Is Soundness sufficient in practice?

The notion of *safeness* (soundness of the reverse inference step, for failure detection after generalization, e.g. for induction) seems to become standard [3, 23, 38, 42]. And in [39, 42] we have also added the notion of *preservation of solutions*. This means that the closing substitutions on the rigid variables of the sub-goals must solve the input theorem's rigid variables, which make sense as placeholders for concrete bounds and side conditions of the theorem which only a proof can tell.

7.6 Conclusion

Although more useful for proof search in classical logic than Hilbert [19] and Natural Deduction calculi [15], sequent [15] and tableau calculi [14] are still not adequate for a synergetic interplay of human proof guidance and automatic proof search [42], which we hope to achieve with matrix calculi such as CoRe [2].

As the automation of proof search will always fail on the lowest logic level from time to time, be aware: *The fine structure and human-orientedness of a calculus does matter in practice!*

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I have to thank the anonymous referees of previous versions of this paper for the useful elements of their critiques.

I would like to thank my co-lecturers for giving the sometimes better and always less exhausting lectures in our course [4], and the students of the course [4] and especially its predecessor [43] (who were the first to suffer from my formalization of the $(\lim +)$ proof) for teaching each other and sharing all those joys of logic. This means that I would like to thank—among others—Serge Autexier, Christoph Benzmüller, Mark Buckley, Dominik Dietrich, Armin Fiedler, Dieter Hutter, Andreas Meier, Martin Pollet, Marvin Schiller, Tobias Schmidt-Samoa, Jörg Siekmann, Werner Stephan, Fabian M. Suchanek, Marc Wagner, and Magdalena Wolska.

Last but not least, I do thank Chad E. Brown very much indeed for giving me the most careful and constructive comments and suggestions for improvement I ever got in my life. A comparison of his report with one of the anonymous ones of 14th Int. Conf. on Tableaus and Related Methods, Koblenz, 2005, (of a previous version of this paper) suggests that new forms of evaluation that further science by communication between scientists are in great demand. I would like to dedicate this paper to Chad, for various reasons.

Hie iſt Weisheit. Wer verſtand hat/der uberlege
— [22, OFFENBARUNG XIII]

Notes

Note 1 A scornful anonymous referee of a previous version of this paper (who was the only one to reject it for the 14th Int. Conf. on Tableaus and Related Methods, Koblenz, 2005) wrote:

> "For once a positive comment: The first lines of page 12 finally contain a very interesting insight, namely that different non-permutabilities can hide each other."

Note 2 Indeed, in [27] we read:

> "ML's execution profiler reported that the sharing mechanism, meant to boost efficiency, was consuming most of the run time. The replacement of structure sharing by copying made ISABELLE simpler and faster. Complex algorithms are often the problem, not the solution."

Note 3 I did not succeed in finding a really satisfying definition of non-local permutability that fits the non-local situation of the failure of the $(\lim +)$ proof as presented in the lecture courses [4, 43]. The problem was to permute the critical β-step from below the critical δ^+-steps to a place far up above the δ^+-steps. And on this partial path from β down to δ^+ there were other inference steps which may or may not contribute to the non-permutability. Thus, instead of globalizing the notion of permutability I localized the example proof; although the original version had pedagogical advantages.

Furthermore, note that it may be possible to demonstrate the permutability problems of the β-rule with slightly smaller artificial examples. But we prefer a practical example to demonstrate the practical difficulties and discuss some less formal soft aspects which may be more important than the hard non-permutability results of this paper. Moreover, because of its many interesting aspects, this proof will be useful as a standard example for further reference. If you are not in love with formal proofs, I do apologize for the inconvenience of my decision and ask you to send me an E-mail of complaint if you will not have learned something that is worth your efforts in the end. If I receive at least three E-mails seriously stating that these efforts were in vain but the non-permutability deserves proper publication, I will try to produce a version of this paper with a somewhat smaller artificial example.

Note 4 An anonymous referee of a previous version of this paper wrote:

> "The arguments against the use of δ^{+^+} (that the proofs found this way are not human-oriented) are not convincing. It is well-known that improved Skolemization rules can be simulated with applications of the cut rule. So one could proceed as follows. Use δ^{+^+} for proof generation, for presentation insert the respective cut steps. This way any forms of sophisticated Skolemization could be replaced by case distinctions, which are easily understandable by any human user."

The point that is missed in this critique is the following. The automatic generation of non-trivial proofs is typically not possible today and probably will never be. Thus, besides some rare exceptions—as the automation of proof search will always fail on the lowest logic level from time to time—the only chance for automatic theorem proving to become useful for mathematicians is a synergetic interplay between the mathematician and the machine. For this interplay—to give the human user a chance to interact—the calculus *itself* must be human-oriented. Thus, it does not suffice to compute human-oriented representations; not in the end, and—as the syntactical problems have to be presented accurately—also not intermediately in a user interface.

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